Hardy and BMO spaces on graphs, application to Riesz transform

Joseph Feneuil

Hardy and BMO spaces on graphs, application to Riesz transform.

Joseph Feneuil
Institut Fourier, Grenoble

January 7, 2015

Abstract

Let Γ be a graph with the doubling property for the volume of balls and P a reversible random walk on Γ. We introduce H^1 Hardy spaces of functions and 1-forms adapted to P and prove various characterizations of these spaces. We also characterize the dual space of H^1 as a BMO-type space adapted to P. As an application, we establish H^1 and H^1-L^1 boundedness of the Riesz transform.

Keywords: Graphs - Hardy spaces - Differential forms - BMO spaces - Riesz transform - Gaffney estimates.

Contents

We use the following notations. $A(x) \lesssim B(x)$ means that there exists C independant of x such that $A(x) \leq C\,B(x)$ for all x, while $A(x) \simeq B(x)$ means that $A(x) \lesssim B(x)$ and $B(x) \lesssim A(x)$. The parameters from which the constant is independant will be either obvious from context or recalled.
Furthermore, if E, F are Banach spaces, $E \subset F$ means that E is continuously included in F. In the same way, $E = F$ means that the norms are equivalent.

1 Introduction and statement of the results

The study of real variable Hardy spaces in $\mathbb{R}^n$ began in the early 1960's with the paper of Stein and Weiss [26]. At the time, the spaces were defined by means of Riesz transforms and harmonic functions. Fefferman and Stein provided in [16] various characterizations (for instance in terms of suitable maximal functions) and developed real variable methods for the study of Hardy spaces.

In several issues in harmonic analysis, $H^1(\mathbb{R}^n)$ turns out to be the proper substitute of $L^1(\mathbb{R}^n)$. For example, the Riesz transforms, namely the operators $R_j = \partial_j(-\Delta)^{-\frac{1}{2}}$, are $L^p(\mathbb{R}^n)$ bounded for all $p \in (1, +\infty)$, $H^1(\mathbb{R}^n)$-bounded, but not $L^1(\mathbb{R}^n)$-bounded (see [22]).

Hardy spaces were defined in the more general context of spaces of homogeneous type by Coifman and Weiss in [8], by means of an atomic decomposition. An atom is defined as a function supported in a ball, with zero integral and suitable size condition. However, even in the Euclidean context, the definition of the Hardy space H^1 given by Coifman and Weiss is not always suited to the H^1-L^1 boundedness of some Calderòn-Zygmund type operators. Indeed, the cancellation condition satisfied by atoms does not always match with differential operators (consider the case of $-\operatorname{div}(A\nabla)$ on $\mathbb{R}^n$, for instance).

To overcome this difficulty, Hardy spaces adapted to operators were developed in various frameworks during the last decade. In 2005, in [14] and [15], Duong and Yan defined Hardy and BMO spaces for an operator L when the kernel of the semigroup generated by L satisfies a pointwise Gaussian upper bound. It was discovered later that, together with the doubling condition for the volumes of balls, L^2 Davies-Gaffney type estimates for the semigroup generated by L are enough to develop a quite rich theory of Hardy spaces on Riemannian manifolds (see [3]) and for second order divergence form elliptic operator in $\mathbb{R}^n$ with measurable complex coefficients (see [21]). These ideas were pushed further in the general context of doubling measure spaces when L is self-adjoint (see [19]).

The present work is devoted to an analogous theory of Hardy spaces in a discrete context, namely in graphs Γ equipped with a suitable discrete Laplace operator, given by $I - P$ where P is a Markov operator (see [18] and the references therein). We define and give various characterizations of the Hardy space $H^1(\Gamma)$ adapted to P, under very weak assumptions on Γ. The first characterization is formulated in terms of quadratic functionals (of Lusin type), relying on results and methods developed in [4] and [17]. The second one is the molecular (or atomic) decomposition of $H^1(\Gamma)$. A description of the dual space of $H^1(\Gamma)$ as a BMO-type space is obtained.

We also deal with the Riesz transform on Γ, namely the operator $d(I - P)^{-\frac{1}{2}}$, where d stands for the differential on Γ (*i.e.* $df(x, y) := f(y) - f(x)$ for all functions f on Γ and all edges (x, y)). When $p \in (1, +\infty)$, the L^p-boundedness of the Riesz transform was dealt with in [4, 24]. Here, we prove an endpoint boundedness result for $p = 1$: roughly speaking, the Riesz transform is H^1-bounded. In the same spirit as [3], this assertion requires the definition a Hardy space of "exact 1-forms" on the edges of Γ. We define and give characterizations of this space by quadratic functionals and molecular decompositions. Finally, the H^1-boundedness of the Riesz transform is established.

Some Hardy spaces associated with $I - P$ were introduced and characterized in [6], together with a description of their duals and the H^1-L^1 boundedness of Riesz transform was proved. Even if the authors in [6] also deal with the case of H^p for $p < 1$, their assumptions on P are stronger than ours (they assume a pointwise Gaussian upper bound on the iterates of the kernel of P, which is not required for most of our results) and they do not consider Hardy spaces of forms. Moreover, the Hardy spaces introduced in the present work are bigger than the ones in [6].

1.1 The discrete setting

Let Γ be an infinite set and $\mu_{xy} = \mu_{yx} \geq 0$ a symmetric weight on $\Gamma \times \Gamma$. The couple (Γ, μ) induces a (weighted unoriented) graph structure if we define the set of edges by

$$E = \{(x, y) \in \Gamma \times \Gamma,\, \mu_{xy} > 0\}.$$

We call then x and y neighbours (or $x \sim y$) if $(x, y) \in E$.
We will assume that the graph is connected and locally uniformly finite. A graph is connected if for all $x, y \in \Gamma$, there exists a path $x = x_0, x_1, \dots, x_N = y$ such that for all $1 \leq i \leq N$, $x_{i-1} \sim x_i$ (the length of such path is then N). A graph is said to be locally uniformly finite if there exists $M_0 \in \mathbb{N}$ such that for all $x \in \Gamma$, $\#\{y \in \Gamma,\, y \sim x\} \leq M_0$ (i.e. the number of neighbours of a vertex is uniformly bounded).

The graph is endowed with its natural metric d, which is the shortest length of a path joining two points. For all $x \in \Gamma$ and all $r > 0$, the ball of center x and radius r is defined as $B(x,r) = \{y \in \Gamma,\, d(x,y) < r\}$. In the opposite way, the radius of a ball B is the only integer r such that $B = B(x_B, r)$ (with x_B the center of B). Therefore, for all balls $B = B(x,r)$ and all $\lambda \geq 1$, we set $\lambda B := B(x, \lambda r)$ and define $C_j(B) = 2^{j+1}B \backslash 2^j B$ for all $j \geq 2$ and $C_1(B) = 4B$.
If $E, F \subset \Gamma$, $d(E,F)$ stands for the distance between E and F, namely

$$d(E,F) = \inf_{x \in E,\ y \in F} d(x,y).$$

We define the weight $m(x)$ of a vertex $x \in \Gamma$ by $m(x) = \sum_{x \sim y} \mu_{xy}$. More generally, the volume of a subset $E \subset \Gamma$ is defined as $m(E) := \sum_{x \in E} m(x)$. We use the notation $V(x,r)$ for the volume of the ball $B(x,r)$, and in the same way, $V(B)$ represents the volume of a ball B.
We define now the $L^p(\Gamma)$ spaces. For all $1 \leq p < +\infty$, we say that a function f on Γ belongs to $L^p(\Gamma, m)$ (or $L^p(\Gamma)$) if

$$\|f\|_p := \left(\sum_{x \in \Gamma} |f(x)|^p m(x) \right)^{\frac{1}{p}} < +\infty,$$

while $L^\infty(\Gamma)$ is the space of functions satisfying

$$\|f\|_\infty := \sup_{x \in \Gamma} |f(x)| < +\infty.$$

Let us define for all $x, y \in \Gamma$ the discrete-time reversible Markov kernel p associated with the measure m by $p(x,y) = \frac{\mu_{xy}}{m(x)m(y)}$. The discrete kernel $p_l(x,y)$ is then defined recursively for all $l \geq 0$ by

$$\begin{cases} p_0(x,y) = \frac{\delta(x,y)}{m(y)} \\ p_{l+1}(x,y) = \sum_{z \in \Gamma} p(x,z) p_l(z,y) m(z). \end{cases} \tag{1}$$

Remark 1.1. *Note that this definition of p_l differs from the one of p_l in [24], [4] or [12], because of the $m(y)$ factor. However, p_l coincides with K_l in [13]. Remark that in the case of the Cayley graphs of finitely generated discrete groups, where $m(x) = 1$ for all x, the definitions coincide.*

Notice that for all $l \geq 1$, we have

$$\|p_l(x,.)\|_{L^1(\Gamma)} = \sum_{y \in \Gamma} p_l(x,y) m(y) = \sum_{d(x,y) \leq l} p_l(x,y) m(y) = 1 \qquad \forall x \in \Gamma, \tag{2}$$

and that the kernel is symmetric:

$$p_l(x,y) = p_l(y,x) \qquad \forall x, y \in \Gamma. \tag{3}$$

For all functions f on Γ, we define P as the operator with kernel p, i.e.

$$Pf(x) = \sum_{y \in \Gamma} p(x,y) f(y) m(y) \qquad \forall x \in \Gamma. \tag{4}$$

It is easily checked that P^l is the operator with kernel p_l.
Since $p(x,y) \geq 0$ and (2) holds, one has, for all $p \in [1, +\infty]$,

$$\|P\|_{p \to p} \leq 1. \tag{5}$$

Remark 1.2. *Let $1 \leq p < +\infty$. Since, for all $l \geq 0$, $\left\|P^l\right\|_{p \to p} \leq 1$, the operators $(I-P)^\beta$ and $(I+P)^\beta$ are L^p-bounded for all $\beta \geq 0$ (see [11]).*

We define a nonnegative Laplacian on Γ by $\Delta = I - P$. One has then

Remark 1.3. *One can check that $\|\Delta\|_{1 \to 1} \leq 2$. Moreover, the previous remark states that Δ^β is $L^1(\Gamma)$-bounded. Note that the L^1-boundedness of the operators Δ^β is not true in the continuous setting (such as Riemannian manifolds), and makes some proofs of the present paper easier than in the case of Riemannian manifolds. In particular, we did not need then to prove similar results of the ones in [2].*

$$
\begin{aligned}
< (I-P)f, f >_{L^2(\Gamma)} &= \sum_{x,y\in\Gamma} p(x,y)(f(x)-f(y))f(x)m(x)m(y) \\
&= \frac{1}{2}\sum_{x,y\in\Gamma} p(x,y)|f(x)-f(y)|^2 m(x)m(y),
\end{aligned}
\tag{6}
$$

where we use (2) for the first equality and (3) for the second one. The last calculus proves that the following operator

$$
\nabla f(x) = \left(\frac{1}{2}\sum_{y\in\Gamma} p(x,y)|f(y)-f(x)|^2 m(y)\right)^{\frac{1}{2}},
$$

called "length of the gradient" (and the definition of which is taken from [9]), satisfies

$$
\|\nabla f\|^2_{L^2(\Gamma)} = < (I-P)f, f >_{L^2(\Gamma)} = \|\Delta^{\frac{1}{2}} f\|_{L^2(\Gamma)}. \tag{7}
$$

1.2 Assumptions on the graph

Definition 1.4. *We say that* (Γ,μ) *satisfies the doubling property if there exists* $C>0$ *such that*

$$
V(x,2r) \leq CV(x,r) \qquad \forall x\in\Gamma,\ \forall r>0. \tag{DV}
$$

Proposition 1.5. *Let* (Γ,μ) *satisfying the doubling property. Then there exists* $d>0$ *such that*

$$
V(x,\lambda r) \lesssim \lambda^d V(x,r) \qquad \forall x\in\Gamma,\ r>0 \text{ and } \lambda\geq 1. \tag{8}
$$

We denote by d_0 *the infimum of the* d *satisfying* (8).

Definition 1.6. *We say that* (Γ,μ) *(or* P*) satisfies* (LB) *if there exists* $\epsilon=\epsilon_{LB}>0$ *such that*

$$
p(x,x)m(x) \geq \epsilon \qquad \forall x\in\Gamma. \tag{LB}
$$

Remark 1.7. *In particular, the condition* (LB) *implies that* -1 *does not belong to the* L^2*-spectrum of* P*, which implies in turn the analyticity of* P *in* $L^p(\Gamma)$*,* $1<p<+\infty$ *([11]).*

From now on, all the graphs considered (unless explicitely stated) satisfy the doubling property and (LB). In this context, Coulhon, Grigor'yan and Zucca proved in [10] (Theorem 4.1) that the following Davies-Gaffney estimate holds:

Theorem 1.8. *Assume that* (Γ,μ) *satisfies* (DV)*. Then there exist* $C,c>0$ *such that for all subsets* $E,F\subset\Gamma$ *and all fonctions* f *supported in* F*, one has*

$$
\|P^{l-1}f\|_{L^2(E)} \leq C\exp\left(-c\frac{d(E,F)^2}{l}\right)\|f\|_{L^2(F)} \qquad \forall l\in\mathbb{N}^*. \tag{GUE}
$$

The estimate (GUE), also called Gaffney estimate, will be sufficient to prove most of the results of this paper. However, some results proven here can be improved if we assume the following stronger pointwise gaussian estimate:

Definition 1.9. *We say that* (Γ,μ) *satisfies* (UE) *if there exist* $C,c>0$ *such that*

$$
p_{l-1}(x,y) \leq C\frac{1}{V(x,\sqrt{l})}\exp\left(-c\frac{d(x,y)^2}{l}\right) \qquad \forall x,y\in\Gamma,\ \forall l\in\mathbb{N}^*. \tag{UE}
$$

Remark 1.10. *Under* (DV)*, property* (UE) *is equivalent to*

$$
p_{l-1}(x,x) \leq \frac{C}{V(x,\sqrt{l})} \qquad \forall x\in\Gamma,\ \forall l\in\mathbb{N}^*. \tag{DUE}
$$

The conjonction of (DV) *and* (UE) *(or* (DUE)*) is also equivalent to some relative Faber-Krahn inequality (see [9]).*

1.3 Definition of Hardy spaces on weighted graphs

We introduce three different definitions for Hardy spaces. The first two ones rely on molecular decomposition.

Definition 1.11. *Let $M \in \mathbb{N}^*$. When $\epsilon \in (0,+\infty)$, a function $a \in L^2(\Gamma)$ is called a (BZ_1, M, ϵ)-molecule if there exist $s \in \mathbb{N}^*$, a M-tuple $(s_1,\dots,s_M) \in [\![s,2s]\!]^M$, a ball B of radius $\sqrt{s}$ and a function $b \in L^2(\Gamma)$ such that*

(i) $a = (I-P^{s_1})\dots(I-P^{s_M})b$,

(ii) $\|b\|_{L^2(C_j(B))} \leq 2^{-j\epsilon} V(2^jB)^{-\frac{1}{2}}$, $\forall j \geq 1$.

A function $a \in L^2(\Gamma)$ is called a (BZ_1, M, ∞)-molecule (or a (BZ_1, M)-atom) if there exist $s \in \mathbb{N}^*$, a M-tuple $(s_1,\dots,s_M) \in [\![1,M]\!]^M$ a ball B of radius $\sqrt{s}$ and a function $b \in L^2(\Gamma)$ supported in B such that

(i) $a = (I-P^{s_1})\dots(I-P^{s_M})b$,

(ii) $\|b\|_{L^2} = \|b\|_{L^2(B)} \leq V(B)^{-\frac{1}{2}}$.

We say that a (BZ_1, M, ϵ)-molecule a is associated with an integer s, a M-tuple $(s_1,\dots,s_M)$ and a ball B when we want to refer to s, $(s_1,\dots,s_M)$ and B given by the definition.

The second kind of molecules we consider are defined via the operators $I-(I+s\Delta)^{-1}$:

Definition 1.12. *Let $M \in \mathbb{N}^*$. When $\epsilon \in (0,+\infty)$, a function $a \in L^2(\Gamma)$ is called a (BZ_2, M, ϵ)-molecule if there exist $s \in \mathbb{N}^*$, a ball B of radius $\sqrt{s}$ and a function $b \in L^2(\Gamma)$ such that*

(i) $a = [I-(I+s\Delta)^{-1}]^M b$,

(ii) $\|b\|_{L^2(C_j(B))} \leq 2^{-j\epsilon} V(2^jB)^{-\frac{1}{2}}$, $\forall j \geq 1$.

A function $a \in L^2(\Gamma)$ is called a (BZ_2, M, ∞)-molecule (or a (BZ_2, M)-atom) if there exist $s \in \mathbb{N}^*$, a ball B of radius $\sqrt{s}$ and a function $b \in L^2(\Gamma)$ supported in B such that

(i) $a = [I-(I+s\Delta)^{-1}]^M b$,

(ii) $\|b\|_{L^2} = \|b\|_{L^2(B)} \leq V(B)^{-\frac{1}{2}}$.

We say that a (BZ_2, M, ϵ)-molecule a is associated with an integer s and a ball B when we want to refer to s and B given by the definition.

Remark 1.13.
1. *When b is the function occurring in Definition 1.11 or in Definition 1.12, note that $\|b\|_{L^2} \lesssim V(B)^{-\frac{1}{2}}$.*
2. *As will be seen in Proposition 2.7 below, when a is a molecule occurring in Definition 1.11 or in Definition 1.12, one has $\|a\|_{L^1} \lesssim 1$.*

Definition 1.14. *Let $M \in \mathbb{N}^*$ and $\kappa \in \{1,2\}$.*

Let $\epsilon \in (0,+\infty]$. We say that f belongs to $H^1_{BZ\kappa,M,\epsilon}(\Gamma)$ if f admits a molecular (BZ_κ, M, ϵ)-representation, that is if there exist a sequence $(\lambda_i)_{i\in\mathbb{N}} \in \ell^1$ and a sequence $(a_i)_{i\in\mathbb{N}}$ of (BZ_κ, M, ϵ)-molecules such that

$$f = \sum_{i=0}^{\infty} \lambda_i a_i \tag{9}$$

where the convergence of the series to f holds pointwise. The space is outfitted with the norm

$$\|f\|_{H^1_{BZ\kappa,M,\epsilon}} = \inf\left\{ \sum_{j=0}^{\infty} |\lambda_j|,\ \sum_{j=0}^{\infty} \lambda_j a_j,\ \textit{is a molecular } (BZ_\kappa, M, \epsilon)\textit{-representation of } f \right\}.$$

Proposition 1.15. *Let $M \in \mathbb{N}^*$ and $\kappa \in \{1,2\}$. Then the space $H^1_{BZ\kappa,M,\epsilon}(\Gamma)$ is complete. Moreover, $H^1_{BZ\kappa,M,\epsilon}(\Gamma) \subset L^1(\Gamma)$.*

Proof: That $H^1_{BZ\kappa,M,\epsilon}(\Gamma) \subset L^1(\Gamma)$ follows at once from assertion 2 in Remark 1.13, which shows that, if $f \in H^1_{BZ\kappa,M,\epsilon}(\Gamma)$, the series (9) converges in $L^1(\Gamma)$, and therefore converges to f in $L^1(\Gamma)$. Moreover, the space $H^1_{BZ\kappa,M,\epsilon}(\Gamma)$ is complete if it has the property

$$\sum_{j=0}^{\infty} \|f_j\|_{H^1_{BZ\kappa,M,\epsilon}} < +\infty \Longrightarrow \sum_{j=0}^{\infty} f_j \text{ converges in } H^1_{BZ\kappa,M,\epsilon}(\Gamma).$$

This fact is a straightforward consequence of the fact that $\|a\|_{L^1} \lesssim 1$ whenever a is a molecule (see Remark 1.13 and Proposition 2.7). See also the argument for the completeness of H^1_L in [21], p. 48. □

Remark 1.16. *The BZ_κ molecules are molecules in the sense of Bernicot and Zhao in [5] (and then BZ_κ are Hardy spaces in the sense of Bernicot and Zhao). Note that the definition of molecules is slightly different from the one given in [3], [21] or [19]. The article [5] provides some properties of the spaces $H^1_{BZ\kappa,M,\epsilon}$. In particular, under the assumption* (UE)*, these Hardy spaces are suited for L^p interpolation (see Remark 1.41 below) .*

The third Hardy space is defined via quadratic functionals.

Definition 1.17. *Define, for $\beta > 0$, the quadratic functionals L_β on $L^2(\Gamma)$ by*

$$L_\beta f(x) = \left(\sum_{(y,l)\in\gamma(x)} \frac{(l+1)^{2\beta-1}}{V(x,\sqrt{l+1})} |\Delta^\beta P^l f(y)|^2 m(y) \right)^{\frac{1}{2}}$$

where $\gamma(x) = \left\{(y,l) \in \Gamma \times \mathbb{N},\, d(x,y)^2 \leq l\right\}$.

Remark 1.18. *One can also use instead of L_β the Lusin functional $\tilde{L}_\beta$ defined by*

$$\tilde{L}_\beta f(x) = \left(\sum_{(y,k)\in\tilde{\gamma}(x)} \frac{1}{(k+1)V(x,k+1)} |(k^2\Delta)^\beta P^{k^2} f(y) m(y)|^2 \right)^{\frac{1}{2}}$$

where $\tilde{\gamma}(x) = \{(y,k) \in \Gamma \times \mathbb{N},\, d(x,y) \leq k\}$.

The functionals L_β and $\tilde{L}_\beta$ are two different ways to discretize the "countinuous" Lusin functional defined by

$$\begin{aligned} L^c_\beta f(x) &= \left(\int_0^\infty \int_{d(y,x)^2<s} \frac{1}{sV(x,\sqrt{s})} |(s\Delta)^\beta e^{-s\Delta} f(y)|^2 d\mu(y)\, ds \right)^{\frac{1}{2}} \\ &= \left(\int_0^\infty \int_{d(y,x)<t} \frac{1}{tV(x,t)} |(t^2\Delta)^\beta e^{-t^2\Delta} f(y)|^2 d\mu(y)\, dt \right)^{\frac{1}{2}}. \end{aligned}$$

Definition 1.19. *The space $E^1_{quad,\beta}(\Gamma)$ is defined by*

$$E^1_{quad,\beta}(\Gamma) := \left\{ f \in L^2(\Gamma),\, \|L_\beta f\|_{L^1} < +\infty \right\}.$$

It is outfitted with the norm

$$\|f\|_{H^1_{quad,\beta}} := \|L_\beta f\|_{L^1}.$$

Remark 1.20. *Notice that $\|f\|_{H^1_{quad,\beta}}$ is a norm because the null space of Δ is reduced to $\{0\}$ (because the set Γ is infinite by assumption). So, if $k > \beta$ is an integer and $f \in L^2(\Gamma)$ is such that $\Delta^\beta f = 0$, then $\Delta^k f = \Delta^{k-\beta}\Delta^\beta f = 0$, so that $f = 0$.*

Remark 1.21. *Replacing L_β by $\tilde{L}_\beta$ in the definition of $E^1_{quad,\beta}$ yields an equivalent space $\tilde{E}^1_{quad,\beta}$, in the sense that the sets are equal and the norms are equivalent. The proof of this nontrivial fact can be done by adapting the proof of Theorem 1.36 below (details are left to the reader).*

1.4 Definition of BMO spaces on weighted graphs

Fix $x_0 \in \Gamma$ and let $B_0 = B(x_0, 1) = \{x_0\}$. For $\epsilon > 0$ and $M \in \mathbb{N}$, for all functions $\phi \in L^2(\Gamma)$ which can be written as $\phi = \Delta^M \varphi$ for some function $\varphi \in L^2$, define

$$\|\phi\|_{\mathcal{M}_0^{M,\epsilon}} := \sup_{j \geq 1} \left[2^{j\epsilon} V(2^j B_0)^{\frac{1}{2}} \|\varphi\|_{L^2(C_j(B_0))} \right] \in [0, +\infty].$$

We set then

$$\mathcal{M}_0^{M,\epsilon} := \left\{ \phi = \Delta^M \varphi \in L^2(\Gamma), \ \|\phi\|_{\mathcal{M}_0^{M,\epsilon}} < +\infty \right\}.$$

Definition 1.22. *For any $M \in \mathbb{N}$, we set,*

$$\mathcal{E}_M = \bigcup_{\epsilon > 0} (\mathcal{M}_0^{M,\epsilon})^*$$

and

$$\mathcal{F}_M = \bigcap_{\epsilon > 0} (\mathcal{M}_0^{M,\epsilon})^*.$$

Proposition 1.23. *Let $M \in \mathbb{N}$, $s \in \mathbb{N}^*$ and $(s_1, \dots, s_M) \in [\![s, 2s]\!]^M$. If $f \in \mathcal{E}_M$, then the functions $(I - P^{s_1}) \dots (I - P^{s_M}) f$ and $(I - (I + s\Delta)^{-1})^M f$ can be defined in the sense of distributions and are included in $L^2_{loc}(\Gamma)$.*

Proof: The proof of this fact is done in Lemma 3.2. □

Definition 1.24. *Let $M \in \mathbb{N}$. Let $f \in \mathcal{E}_M$.*

We say then that f belongs to $BMO_{BZ1,M}(\Gamma)$ if

$$\|f\|_{BMO_{BZ1,M}} := \sup_{\substack{s \in \mathbb{N}^*, \\ (s_1, \dots, s_M) \in [\![s, 2s]\!]^M, \\ B \text{ of radius } \sqrt{s}}} \left(\frac{1}{V(B)} \sum_{x \in B} |(I - P^{s_1}) \dots (I - P^{s_M}) f(x)|^2 m(x) \right)^{\frac{1}{2}} < +\infty. \tag{10}$$

We say then that f belongs to $BMO_{BZ2,M}(\Gamma)$ if

$$\|f\|_{BMO_{BZ2,M}} := \sup_{\substack{s \in \mathbb{N}^*, \\ B \text{ of radius } \sqrt{s}}} \left(\frac{1}{V(B)} \sum_{x \in B} |[I - (I + s\Delta)^{-1}]^M f(x)|^2 m(x) \right)^{\frac{1}{2}} < +\infty. \tag{11}$$

1.5 Definition of Hardy spaces of 1-forms

We define, for all $x \in \Gamma$, the set $T_x = \{(x, y) \in \Gamma^2, \, y \sim x\}$ and

$$T_\Gamma = \bigcup_{x \in \Gamma} T_x = \{(x, y) \in \Gamma^2, \, y \sim x\}.$$

Definition 1.25. *If $x \in \Gamma$, we define, for all F_x defined on T_x the norm*

$$\|F_x\|_{T_x} = \left(\frac{1}{2} \sum_{y \sim x} p(x, y) m(y) |F_x(x, y)|^2 \right)^{\frac{1}{2}}.$$

Moreover, a function $F : T_\Gamma \to \mathbb{R}$ belongs to $L^p(T_\Gamma)$ if

(i) F is antisymmetric, that is $F(x, y) = -F(y, x)$ for all $x \sim y$,

(ii) $\|F\|_{L^p(T_\Gamma)} < +\infty$, with

$$\|F\|_{L^p(T_\Gamma)} = \|x \mapsto \|F(x, .)\|_{T_x}\|_{L^p(\Gamma)}.$$

Definition 1.26. *Let $f : \Gamma \to \mathbb{R}$ and $F : T_\Gamma \to \mathbb{R}$ be some functions. Define the operators d and d^* by*

$$df(x, y) := f(x) - f(y) \qquad \forall (x, y) \in T_\Gamma$$

and

$$d^* F(x) := \sum_{y \sim x} p(x, y) F(x, y) m(y) \qquad \forall x \in \Gamma.$$

Remark 1.27. *It is plain to see that $d^*d = \Delta$ and $\|df(x,.)\|_{T_x} = \nabla f(x)$.*

The definition of Hardy spaces of 1-forms is then similar to the case of functions. First, we introduce Hardy spaces via molecules.

Definition 1.28. *Let $M \in \mathbb{N}$ and $\epsilon \in (0,+\infty)$. A function $a \in L^2(T_\Gamma)$ is called a $(BZ_2, M+\frac{1}{2}, \epsilon)$-molecule if there exist $s \in \mathbb{N}^*$, a ball B of radius $\sqrt{s}$ and a function $b \in L^2(\Gamma)$ such that*

(i) $a = s^{M+\frac{1}{2}} d\Delta^M (I - s\Delta)^{-M-\frac{1}{2}} b$;

(ii) $\|b\|_{L^2(C_j(B))} \leq 2^{-j\epsilon} V(2^j B)^{-\frac{1}{2}}$ for all $j \geq 1$.

Remark 1.29. *As in the case of functions, Corollary 2.12 below implies a uniform bound on the L^1 norm of molecules, that is, for all $M \in \mathbb{N}$ and all $\epsilon \in (0,+\infty)$, there exists $C >$ such that each (BZ_2, M, ϵ)-molecule a satisfies*

$$\|a\|_{L^1(T_\Gamma)} \leq C.$$

Definition 1.30. *Let $M \in \mathbb{N}$ and $\epsilon \in (0,+\infty)$. We say that F belongs to $H^1_{BZ2,M+\frac{1}{2},\epsilon}(T_\Gamma)$ if F admits a molecular $(BZ_2, M+\frac{1}{2}, \epsilon)$-representation, that is if there exist a sequence $(\lambda_i)_{i\in\mathbb{N}} \in \ell^1$ and a sequence $(a_i)_{i\in\mathbb{N}}$ of $(BZ_2, M+\frac{1}{2}, \epsilon)$-molecules such that*

$$F = \sum_{i=0}^{\infty} \lambda_i a_i$$

where the sum converges pointwise on T_Γ . The space is outfitted with the norm

$$\|f\|_{H^1_{BZ2,M+\frac{1}{2},\epsilon}} = \inf \left\{ \sum_{i=0}^{\infty} |\lambda_i|, \ \sum_{i=0}^{\infty} \lambda_i a_i \text{ is a molecular } (BZ_2, M + \frac{1}{2}, \epsilon)\text{-representation of } f \right\}.$$

Remark 1.31. *The space $H^1_{BZ2,M+\frac{1}{2},\epsilon}(T_\Gamma)$ is complete. The argument is analogous to the one of Proposition 1.15.*

In order to define the Hardy spaces of forms associated with operators, we introduce the L^2 adapted Hardy spaces $H^2(T_\Gamma)$ defined as the closure in $L^2(T_\Gamma)$ of

$$E^2(T_\Gamma) := \{F \in L^2(T_\Gamma), \, \exists f \in L^2(\Gamma) : \, F = df\}.$$

Notice that $d\Delta^{-1}d^* = Id_{E^2(T_\Gamma)}$. The functional $d\Delta^{-1}d^*$ can be extended to a bounded operator on $H^2(T_\Gamma)$ and

$$d\Delta^{-1}d^* = Id_{H^2(T_\Gamma)}. \tag{12}$$

Proposition 1.32. *For all $p \in [1,+\infty]$, the operator d^* is bounded from $L^p(T_\Gamma)$ to $L^p(\Gamma)$.*
The operator $d\Delta^{-\frac{1}{2}}$ is an isometry from $L^2(\Gamma)$ to $L^2(T_\Gamma)$ (or $H^2(T_\Gamma)$), and the operator $\Delta^{-\frac{1}{2}}d^*$ is an isometry from $H^2(T_\Gamma)$ to $L^2(\Gamma)$.

Proof: First, the L^p-boundedness of d^* is provided by

$$\begin{aligned}
\|d^*F\|^p_{L^p(\Gamma)} &= \sum_{x\in\Gamma} \left| \sum_{y\in\Gamma} p(x,y)m(y)F(x,y) \right|^p m(x) \\
&\lesssim \sum_{N^*x\in\Gamma} \|F(x,.)\|^p_{T_x} m(x) = \|F\|^p_{L^p(T_\Gamma)}.
\end{aligned}$$

The L^2-boundedness of $d\Delta^{-\frac{1}{2}}$ is obtained by the calculus

$$\begin{aligned}
\|d\Delta^{-\frac{1}{2}} f\|^2_{L^2(T_\Gamma)} &= \frac{1}{2} \sum_{x\sim y} p(x,y) |\Delta^{-\frac{1}{2}} f(x) - \Delta^{-\frac{1}{2}} f(y)|^2 m(x)m(y) \\
&= \|\nabla \Delta^{-\frac{1}{2}} f\|^2_{L^2(\Gamma)} = \|\Delta^{\frac{1}{2}} \Delta^{-\frac{1}{2}} f\|^2_{L^2(\Gamma)} \\
&= \|f\|^2_{L^2(\Gamma)}.
\end{aligned}$$

The L^2-boundedness of $\Delta^{-\frac{1}{2}}d^*$ is then a consequence of (12). Indeed, if $F \in H^2(\Gamma)$,

$$\begin{aligned}
\|\Delta^{-\frac{1}{2}} d^* F\|_{L^2(\Gamma)} &= \|d\Delta^{-\frac{1}{2}} \Delta^{-\frac{1}{2}} d^* F\|_{L^2(T_\Gamma)} \\
&= \|F\|_{L^2(T_\Gamma)}.
\end{aligned}$$

□

Definition 1.33. *The space $E^1_{quad,\beta}(T_\Gamma)$ is defined by*

$$E^1_{quad,\beta}(T_\Gamma) := \left\{ F \in H^2(T_\Gamma),\ \|L_\beta[\Delta^{-\frac{1}{2}} d^* F]\|_{L^1} < +\infty \right\}$$

equipped with the norm

$$\|F\|_{H^1_{quad,\beta}} := \|L_\beta[\Delta^{-\frac{1}{2}} d^* F]\|_{L^1}.$$

Note that, if $\|F\|_{H^1_{quad,\beta}} = 0$, one has $\Delta^{-1/2} d^* F = 0$, so that $d\Delta^{-1/2}\Delta^{-1/2} d^* F = 0$, which implies that $F = 0$ since $F \in H^2(T_\Gamma)$. Moreover, check that for all $F \in H^2(T_\Gamma)$, $\|F\|_{H^1_{quad,\beta}} = \|\Delta^{-\frac{1}{2}} d^* F\|_{H^1_{quad,\beta}}$

1.6 Main results

In the following results, Γ is assumed to satisfy (DV) and (LB).

Theorem 1.34. *Let $M \in \mathbb{N}^*$. Then $BMO_{BZ1,M}(\Gamma) = BMO_{BZ2,M}(\Gamma)$.*

Theorem 1.35. *Let $M \in \mathbb{N}^*$ and $\kappa \in \{1,2\}$. Let $\epsilon \in (0,+\infty]$.*

Then the dual space of $H^1_{BZ\kappa,\epsilon}(\Gamma)$ is $BMO_{BZ1,M}(\Gamma) = BMO_{BZ2,M}(\Gamma)$. In particular, the spaces $H^1_{BZ\kappa,M,\epsilon}(\Gamma)$ depend neither on ϵ nor on κ.

Moreover, $BMO_{BZ\kappa}(\Gamma)$, initially defined as a subspace of $\mathcal{E}_M$, is actually included in $\mathcal{F}_M$.

Theorem 1.36. *Let $\beta > 0$ and $\kappa \in \{1,2\}$. The completion $H^1_{quad,\beta}(\Gamma)$ of $E^1_{quad,\beta}(\Gamma)$ in $L^1(\Gamma)$ exists. Moreover, if $M \in (\frac{d_0}{4},+\infty) \cap \mathbb{N}^*$ and $\epsilon \in (0,+\infty]$, then the spaces $H^1_{BZ1,M,\epsilon}(\Gamma)$, $H^1_{BZ2,M,\epsilon}(\Gamma)$ and $H^1_{quad,\beta}(\Gamma)$ coincide. More precisely, we have*

$$E^1_{quad,\beta}(\Gamma) = H^1_{BZ\kappa,M,\epsilon}(\Gamma) \cap L^2(\Gamma).$$

Once the equality $H^1_{BZ1,M,\epsilon}(\Gamma) = H^1_{BZ2,M,\epsilon}(\Gamma) = H^1_{quad,\beta}(\Gamma)$ is established, this space will be denoted by $H^1(\Gamma)$.

Corollary 1.37. *Let $M_1, M_2 > \frac{d_0}{4}$. Then we have the equality*

$$BMO_{BZ1,M_1}(\Gamma) = BMO_{BZ2,M_2}(\Gamma).$$

Theorem 1.38. *Let $\beta > 0$. The completion $H^1_{quad,\beta}(T_\Gamma)$ of $E^1_{quad,\beta}(T_\Gamma)$ in $L^1(T_\Gamma)$ exists.*

Moreover, if $M \in (\frac{d_0}{4} - \frac{1}{2},+\infty) \cap \mathbb{N}$ and $\epsilon \in (0,+\infty)$, then the spaces $H^1_{BZ2,M+\frac{1}{2},\epsilon}(T_\Gamma)$ and $H^1_{quad,\beta}(T_\Gamma)$ coincide. More precisely, we have

$$E^1_{quad,\beta}(T_\Gamma) = H^1_{BZ2,M+\frac{1}{2},\epsilon}(T_\Gamma) \cap L^2(T_\Gamma).$$

Again, the space $H^1_{BZ2,M+\frac{1}{2},\epsilon}(T_\Gamma) = H^1_{quad,\beta}(T_\Gamma)$ will be denoted by $H^1(T_\Gamma)$.

Theorem 1.39. *For this theorem only, assume furthermore that (Γ,μ) satisfies* (UE). *Then M can be choosen arbitrarily in $\mathbb{N}^*$ in Theorem 1.36 and Corollary 1.37, M can be choosen arbitrarily in $\mathbb{N}$ in Theorem 1.38.*

Theorem 1.40. *The Riesz transform $d\Delta^{-\frac{1}{2}}$ is bounded from $H^1(\Gamma)$ to $H^1(T_\Gamma)$. As a consequence the Riesz transform $\nabla\Delta^{-\frac{1}{2}}$ is bounded from $H^1(\Gamma)$ to $L^1(\Gamma)$.*

Proof: By definition,

$$\|d\Delta^{-\frac{1}{2}} f\|_{H^1(T_\Gamma)} \simeq \|d\Delta^{-\frac{1}{2}} f\|_{H^1_{quad,1}(T_\Gamma)} = \|\Delta^{-\frac{1}{2}} d^* d\Delta^{-\frac{1}{2}} f\|_{H^1_{quad,1}(\Gamma)} = \|f\|_{H^1_{quad,1}(\Gamma)} \simeq \|f\|_{H^1(\Gamma)}.$$

Therefore, $d\Delta^{-\frac{1}{2}}$ is H^1-bounded. Moreover, $\|\nabla\Delta^{-\frac{1}{2}} f\|_{L^1(\Gamma)} = \|d\Delta^{-\frac{1}{2}} f\|_{L^1(T_\Gamma)} \lesssim \|d\Delta^{-\frac{1}{2}} f\|_{H^1(T_\Gamma)}$. Indeed, the uniform L^1-bound of $(BZ2, M+\frac{1}{2},\epsilon)$-molecules (see Corollary 2.12) yields

$$H^1(T_\Gamma) = H^1_{BZ2,M+\frac{1}{2},\epsilon}(T_\Gamma) \hookrightarrow L^1(T_\Gamma)$$

for any $M > \frac{d_0}{4} - \frac{1}{2}$. □

Remark 1.41.

(a) *It is easily checked that under* (UE), *the Hardy space $H^1(\Gamma) = H^1_{BZ2,1,\infty}(\Gamma)$ satisfies the assumption of Theorem 5.3 in [5]. As a consequence, the interpolation between $H^1(\Gamma)$ and $L^2(\Gamma)$ provides the spaces $L^p(\Gamma)$, $1 < p < 2$. Together with Theorem 1.40, we can recover the main result of [24], that is: under* (UE), *the Riesz transform $\nabla\Delta^{-\frac{1}{2}}$ is L^p-bounded for all $p \in (1,2]$.*

(b) *An interesting byproduct of Theorem 1.36 is the equality, for any* $\epsilon \in (0,+\infty]$ *and any* $M > \frac{d_0}{4}$*, between the spaces* $H^1_{BZ\kappa,M,\epsilon}(\Gamma) \cap L^2(\Gamma)$ *and* $E^1_{BZ\kappa,M,\epsilon}(\Gamma)$ *defined by*

$$E^1_{BZ\kappa,M,\epsilon}(\Gamma) := \left\{ f \in L^2(\Gamma),\ \sum_{j=0}^{\infty} \lambda_j a_j \text{ is a molecular } (BZ_\kappa, M, \epsilon)\text{-representation of } f \text{ and the series converges in } L^2(\Gamma) \right\}$$

and outfitted with the norm

$$\|f\|_{E^1_{BZ\kappa,M,\epsilon}} = \inf \left\{ \sum_{i\in\mathbb{N}} |\lambda_i|,\ \sum_{j=0}^{\infty} \lambda_j a_j \text{ is a molecular } (BZ_\kappa, M, \epsilon)\text{-representation of } f \text{ and the series converges in } L^2(\Gamma) \right\}.$$

We have similar byproducts of Theorems 1.38 and 1.39. Precise statements and proofs are done in Corollary 4.13. As a consequence, the completion of $E^1_{BZ\kappa,M,\epsilon}(\Gamma)$ *in* $L^1(\Gamma)$ *exists and is equal to* $H^1_{BZ\kappa,M,\epsilon}$*. On Riemannian manifolds or in more general contexts, the proof of this fact is much more complicated and is the main result of [2]. Let us emphasize that the proofs of our main results does not go through the* $E^1_{BZ\kappa,M,\epsilon}$ *spaces.*

(c) *We may replace (i) in the definition of* (BZ_2, M, ϵ)*-molecules by*

(i') $a = (I - (I + s_1\Delta)^{-1}) \dots ((I - (I + s_M\Delta)^{-1})b$, where $(s_1, \dots, s_M) \in [s, 2s]^M$

or

(i") $a = (I - (I + r^2\Delta)^{-1})^M b$, where r is the radius of the ball B (or the smallest integer greater than $\sqrt{s}$)

and still get the same space $H^1_{BZ2,M,\epsilon}(\Gamma)$*.*

(d) *However, when* $M \geq 3$*, it is unclear whether replacing item* (i) *of the definition of* (BZ_1, M, ϵ)*-molecules by*

(i') $a = (I - P^s)^M$

yields the same space $H^1_{BZ1,M,\epsilon}(\Gamma)$*.*

Section 2 is devoted to the proof of auxiliary results that will be useful for the next sections. The proof of Theorem 1.34 is treated in paragraph 3.2 and the proof of Theorem 1.35 is done in paragraph 3.3. In the last section, we establish Theorems 1.36, 1.38 and 1.39.

1.7 Comparison with other papers

- **Comparison with [3]**: In [3], the authors proved analogous results (that is the H^1 boundedness of the Riesz transform under very weak assumptions and the various characterizations of H^1) on Riemannian manifolds. Some differences between the two papers can be noted. First, BMO spaces are not considered there. They also choose to define some Hardy spaces via tent spaces (while we prefer to use Lusin functionals). Contrary to us, they introduced the spaces H^p, for all $p \in [1, +\infty]$, and proved that these spaces form an interpolation scale for the complex method.

- **Comparison with [19]**: This article develops Hardy and BMO spaces adapted to a symmetric operator L in a general context of doubling measure spaces when the semigroup generated by L satisfies L^2 Gaffney estimates. However, on graphs, it is unclear whether these L^2 Gaffney estimates for the semigroup generated by the Laplacian hold or not. Yet, Coulhon, Grigor'yan and Zucca proved in [10] that we have L^2 Gaffney type estimates for the discrete iterates of Markov operators and we only rely on these estimates in the present paper.

- **Comparison with [6]**: First of all, as in [19], there are no results about Hardy spaces on 1-forms and the authors do not prove the H^1 boundedness of the Riesz transforms. Then, as said in the introduction, they assume in all their paper a pointwise gaussian bound of the Markov kernel while it is not required for most of our results. Moreover, the results of the present paper stated under (UE) are stronger that those stated in [6]. Indeed, in the results stated in [6], the constant M need to be greater than $\frac{d_0}{2}$ while, in the present paper, we used the pointwise gaussian bound in order to get rid of the dependance of M on the "dimension" d_0.

  Besides, the definitions of their Hardy spaces and ours *a priori* differ. Let us begin with the Hardy spaces defined via molecules. For convenience, we introduce a new definition of molecules.

  Definition 1.42. *Let* $M \in \mathbb{N}^*$ *and* $\epsilon \in (0, +\infty)$*. A function* $a \in L^2(\Gamma)$ *is called a* (HM, M, ϵ)*-molecule if there exist a ball* B *of radius* $r \in \mathbb{N}^*$ *and a function* $b \in L^2(\Gamma)$ *such that*

  (i) $a = [r^2\Delta]^M b$,

(ii) $\|[r^2\Delta]^k b\|_{L^2(C_j(B))} \leq 2^{-j\epsilon} V(2^j B)^{-\frac{1}{2}}$, $\forall j \in \mathbb{N}^*$, $\forall k \in [\![0, M]\!]$.

The space $H^1_{HM,M,\epsilon}(\Gamma)$ is then defined in the same way as $H^1_{BZ\kappa,M,\epsilon}(\Gamma)$.

Using methods developed in [19] and in the present paper, it can be proved that, if $M > \frac{d_0}{4}$ (or if $M \in \mathbb{N}^*$ if we assume the extra condition (UE)), there is equality between the spaces $H^1_{HM,M,\epsilon}(\Gamma)$ and $H^1_{quad,1}(\Gamma) = H^1(\Gamma)$. The proofs are similar to those of the present paper. The molecules introduced by Bui and Duong - we call them (BD, M, ϵ)-molecules - are the (HM, M, ϵ)-molecules where we replaced r^2 by r in (i) and (ii). It is easily checked that a (BD, M, ϵ)-molecule is a (HM, M, ϵ)-molecule and hence, under assumption (UE), our Hardy spaces are bigger than theirs.

Since they proved (as we do here) that Hardy spaces defined with molecules and with quadratic functionals coincide, the Hardy spaces via quadratic functionals in [6] are also different from ours. Indeed, our Hardy spaces are of parabolic type (heat kernel) while those of [6] are modelled on the Poisson semigroup. Furthermore, they only consider one Lusin functional, while we consider a family of Lusin functionals (indexed by $\beta > 0$), and the independance of Hardy spaces $H^1_{quad,\beta}(\Gamma)$ with respect to β is a key point of the proof of the boundedness of Riesz transforms.

Acknowledgements: the author is grateful to E. Russ for comments and suggestions that improved the paper. He would also like to thank P. Auscher and A. Morris for interesting discussions.

2 Preliminary results

2.1 L^2-convergence

Proposition 2.1. *Let $\beta > 0$. Let P satisfying* (LB). *One has the following convergence: for all $f \in L^2(\Gamma)$,*

$$\sum_{k=0}^{N} a_k (I-P)^\beta P^k f \xrightarrow{N \to +\infty} f \qquad \text{in } L^2(\Gamma)$$

where $\sum a_k z^k$ is the Taylor series of the function $(1-z)^{-\beta}$.

Remark 2.2. *This result extends Lemma 1.13 in [4]. It provides a discrete version of the identity*

$$f = c_\beta \int_0^\infty (t\Delta)^\beta e^{-t\Delta} f dt.$$

Corollary 2.3. *Let (Γ, μ) a weighted graph. One has the following convergence: for all $f \in L^2(\Gamma)$,*

$$\sum_{k=0}^{N} a_k (I-P^2)^\beta P^{2k} f \xrightarrow{N \to +\infty} f \qquad \text{in } L^2(\Gamma)$$

Proof: (Proposition 2.1)

First, notice that Corollary 2.3 is an immediate consequence of Proposition 2.1 since P^2 is a Markov operator satisfying (LB) (see [10]).

Let $f \in L^2(\Gamma)$. Let us check the behavior of

$$\left\| \left[\sum_{k=0}^{N} a_k (I-P)^\beta P^k - I \right] f \right\|_{L^2} \tag{13}$$

when $N \to +\infty$. Since $\|P\|_{2\to 2} = 1$ and P satisfies (LB), there exists $a > -1$ such that

$$P = \int_a^1 \lambda dE(\lambda).$$

Thus

$$\left\| \left[\sum_{k=0}^{N} a_k (I-P)^\beta P^k - I \right] f \right\|_{L^2}^2 = \int_a^1 \left[\sum_{k=0}^{N} a_k (1-\lambda)^\beta \lambda^k - 1 \right]^2 dE_{ff}(\lambda). \tag{14}$$

However,

$$\sum_{k=0}^{N} a_k(1-\lambda)^\beta \lambda^k \xrightarrow{N\to\infty} \begin{cases} 1 & \text{for all } \lambda \in [a,1) \\ 0 & \text{if } \lambda = 1 \end{cases}$$

and since the sum is nonnegative and increasing in N, then

$$\left|\sum_{k=0}^{N} a_k(1-\lambda)^\beta \lambda^k - 1\right| \leq 1 \qquad \forall \lambda \in [a,1].$$

We use this result in (14) to get the uniform bound

$$\left\|\left[\sum_{k=0}^{N} a_k(I-P)^\beta P^k - I\right] f\right\|_{L^2}^2 \leq \int_a^1 dE_{ff}(\lambda) = \|f\|_{L^2}^2. \tag{15}$$

Let us focus on (13) when we furthermore assume that $f \in \mathcal{R}(\Delta)$, that is $f = \Delta g$ for some $g \in L^2(\Gamma)$. The identity (14) reads as

$$\left\|\left[\sum_{k=0}^{N} a_k(I-P)^\beta P^k - I\right] f\right\|_{L^2}^2 = \int_a^1 \left[\sum_{k=0}^{N} a_k(1-\lambda)^{\beta+1}\lambda^k - (1-\lambda)\right]^2 dE_{gg}(\lambda).$$

Yet, $\sum_{k=0}^{N} a_k(1-\lambda)^{\beta+1}\lambda^k - (1-\lambda)$ converges uniformly to 0 for all $\lambda \in [a,1]$.
Consequently, for all $\epsilon > 0$, there exists N_0 such that, for all $N > N_0$,

$$\left\|\left[\sum_{k=0}^{N} a_k(I-P)^\beta P^k - I\right] f\right\|_{L^2}^2 \leq \epsilon \int_a^1 dE_{gg}(\lambda) = \epsilon\|g\|_{L^2}^2.$$

This implies

$$\sum_{k=0}^{N} a_k(I-P)^\beta P^k f \xrightarrow{N\to\infty} f \quad \text{ in } L^2 \text{ and for all } f \in \mathcal{R}(\Delta). \tag{16}$$

Since $L^2 = \overline{\mathcal{R}(\Delta)}$, the combination of (15) and (16) provides the desired conclusion. Indeed, (16) provides the L^2-convergence on the dense space $\mathcal{R}(\Delta)$ and the uniform boundedness (16) allows us to extend the convergence to $L^2(\Gamma)$. □

2.2 Davies-Gaffney estimates

Definition 2.4. *We say that a family of operators $(A_s)_{s\in\mathbb{N}}$ satisfies Davies-Gaffney estimates if there exist three constants $C, c, \eta > 0$ such that for all subsets $E, F \subset \Gamma$ and all functions f supported in F, there holds*

$$\|A_s f\|_{L^2(E)} \leq C \exp\left(-c\left[\frac{d(E,F)^2}{s}\right]^\eta\right)\|f\|_{L^2}. \tag{17}$$

Hofmann and Martell proved in [20, Lemma 2.3] the following result about Davies-Gaffney estimates:

Proposition 2.5. *If A_s and B_t satisfy Davies-Gaffney estimates, then there exist $C, c, \eta > 0$ such that for all subsets $E, F \subset \Gamma$ and all functions f supported in F, there holds*

$$\|A_s B_t f\|_{L^2(E)} \leq C \exp\left(-c\left[\frac{d(E,F)^2}{s+t}\right]^\eta\right)\|f\|_{L^2} \tag{18}$$

In particular, $(A_sB_s)_{s\in\mathbb{N}}$ satisfies Davies-Gaffney estimates.

More precisely, if η_A and η_B are the constants involved in (17) *respectively for A_s and B_t , then the constant η that occurs in* (18) *can be choosen equal to* $\min\{\eta_A, \eta_B\}$.

Proposition 2.6. *Let $M \in \mathbb{N}$. The following families of operators satisfy the Davies-Gaffney estimates* *(i)* $\displaystyle\prod_{i=1}^{M}\left(\frac{1}{t_s^i}\sum_{k=0}^{t_s^i}P^k\right)$, *where for all* $i\in[\![1,M]\!]$, $t_s^i\in[\![1,2s]\!]$,

(ii) $\displaystyle\prod_{i=1}^{M}(I-P^{t_s^i})$, *where for all* $i\in[\![1,M]\!]$, $t_s^i\in[\![s,2s]\!]$,

(iii) $(s\Delta)^M P^s$,

(iv) $(I+s\Delta)^{-M}$,

(v) $(I-(I+s\Delta)^{-1})^M=(s\Delta)^M(I+s\Delta)^{-M}$.

In (i), (ii) and (iii), the parameter η *is equal to 1 and in (iv) and (v),* η *is equal to* $\frac{1}{2}$.

Proof: (i) and (ii) are direct consequences of (GUE) and Proposition 2.5. Assertion (iii) is the consequence of (GUE) and (LB) and a proof can be found in [13].

We turn now to the proof of (iv) and (v). According to Proposition 2.5, it remains to show the Davies-Gaffney estimates for $(I+s\Delta)^{-1}$, and since $s\Delta(I+s\Delta)^{-1}=I-(I+s\Delta)^{-1}$, it is enough to deal with $(I+s\Delta)^{-1}$. The L^2-functional calculus provides the identity

$$\begin{aligned}(I+s\Delta)^{-1}f&=\frac{1}{1+s}\left(I-\frac{s}{1+s}P\right)^{-1}f\\&=\sum_{k=0}^{+\infty}\frac{1}{1+s}\left(\frac{s}{1+s}\right)^kP^kf,\end{aligned}\tag{19}$$

where the convergence holds in $L^2(\Gamma)$.

Let f be a function supported in F. Then, one has with the Gaffney-Davies estimates (GUE):

$$\begin{aligned}\|(I+s\Delta)^{-1}f\|_{L^2(E)}&\lesssim\sum_{k=0}^{+\infty}\frac{1}{1+s}\left(\frac{s}{1+s}\right)^k\|P^kf\|_{L^2(E)}\\&\lesssim\sum_{k=0}^{+\infty}\frac{1}{1+s}\left(\frac{s}{1+s}\right)^k\exp\left(-c\frac{d(E,F)^2}{1+k}\right)\|f\|_{L^2(F)}\\&\lesssim\|f\|_{L^2(F)}\left[\sum_{k=0}^{s}\frac{1}{1+s}\exp\left(-c\frac{d(E,F)^2}{1+k}-c'\frac{k}{1+s}\right)\right.\\&\qquad\left.+\sum_{k=s}^{+\infty}\frac{1+s}{(1+k)^2}\exp\left(-c\frac{d(E,F)^2}{1+k}-c'\frac{k}{1+s}\right)\right].\end{aligned}$$

Yet, the function $\psi:k\in\mathbb{R}^+\mapsto c\frac{d(E,F)^2}{1+k}+c'\frac{k}{1+s}$ is bounded from below and

$$\psi(k)\gtrsim\frac{d(E,F)}{\sqrt{1+s}}.$$

Hence, the use of Lemma B.1 proved in the appendix yields

$$\begin{aligned}\|(I+s\Delta)^{-1}f\|_{L^2(E)}&\lesssim\|f\|_{L^2(F)}\exp\left(-c\frac{d(E,F)}{\sqrt{1+s}}\right)\left[\sum_{k=0}^{s}\frac{1}{1+s}+\sum_{k=s}^{+\infty}\frac{1+s}{(1+k)^2}\right]\\&\lesssim\|f\|_{L^2(F)}\exp\left(-c\frac{d(E,F)}{\sqrt{1+s}}\right).\end{aligned}$$

□

Proposition 2.7. *Let* $\kappa\in\{1,2\}$. *Let* a *be a* (BZ_κ,M,ϵ)*-molecule. Then*

$$\|a\|_{L^1}\lesssim1\qquad\text{and}\qquad\|a\|_{L^2(C_j(B))}\lesssim\frac{2^{-j\epsilon}}{V(2^jB)^{\frac12}}\quad\forall j\in\mathbb{N}^*.$$

Proof: We will only prove the case where $\kappa = 1$. The case $\kappa = 2$ is proven similarly and will therefore be skipped.

Since

$$\|a\|_{L^1} \leq \sum_{j\geq 1} V(2^{j+1}B)^{\frac{1}{2}} \|a\|_{L^2(C_j(B))},$$

we only need to check the second fact. Let $s \in \mathbb{N}$, $(s_1, \dots, s_M) \in [\![s, 2s]\!]^M$ and a ball B associated with the molecule a.

Define $\tilde{C}_j(B) = \bigcup_{k=j-1}^{j+1} C_k(B)$ and observe that $d(C_j(B), \Gamma \backslash \tilde{C}_j(B)) \gtrsim 2^j \sqrt{s}$. Then Proposition 2.6 provides

$$\begin{aligned}
\|a\|_{L^2(C_j(B))} &\leq \|(I-P^{s_1})\dots(I-P^{s_M})[b\mathbb{1}_{\tilde{C}_j(B)}]\|_{L^2(C_j(B))} + \|(I-P^{s_1})\dots(I-P^{s_M})[b\mathbb{1}_{\Gamma\backslash\tilde{C}_j(B)}]\|_{L^2(C_j(B))} \\
&\lesssim \|b\|_{L^2(\tilde{C}_j(B))} + e^{-c4^j}\|b\|_{L^2} \\
&\lesssim \frac{2^{-j\epsilon}}{V(2^jB)^{\frac{1}{2}}} + \frac{e^{-c4^j}}{V(B)^{\frac{1}{2}}} \\
&\lesssim \frac{2^{-j\epsilon}}{V(2^jB)^{\frac{1}{2}}}.
\end{aligned}$$

□

2.3 Gaffney estimates for the gradient

Proposition 2.8. *Let (Γ, μ) satisfying (LB) (note that (DV) is not assumed here). Let $c > 0$ such that*

$$\frac{8ce^{8c}}{\epsilon_{LB}} \leq 1. \tag{20}$$

There exists $C > 0$ such that for all subsets $F \subset \Gamma$ and all f supported in F, one has

$$\left\| P^k f e^{c\frac{d^2(.,F)}{k+1}} \right\|_{L^2} \leq C\|f\|_{L^2}.$$

The proof of Proposition 2.8 is based on the following result of Coulhon, Grigor'yan and Zucca:

Lemma 2.9. *Let (Γ, μ) satisfying (LB). Let $(k, x) \mapsto g_k(x)$ be a positive function on $\mathbb{N} \times \Gamma$. Then, for all finitely supported functions $f \in L^2(\Gamma)$ and for all $k \in \mathbb{N}$,*

$$\left\|\sqrt{g_{k+1}}P^{k+1}f\right\|_{L^2}^2 - \left\|\sqrt{g_k}P^kf\right\|_{L^2}^2 \leq \sum_{x\in\Gamma} |P^kf(x)|^2 \left(g_{k+1}(x) - g_k(x) + \frac{|\nabla g_{k+1}(x)|^2}{4\epsilon_{LB}g_{k+1}(x)}\right) m(x)$$

Proof: This fact is actually established in the proof of [10, Theorem 2.2, pp. 566-567]. □

Proof: (Proposition 2.8).

First, let us prove the result for f supported in a finite set $F \subset \Gamma$. Let f (finitely supported and) supported in F. We wish to use Lemma 2.9 with

$$g_k(x) = e^{2c\frac{d^2(x,F)}{k+1}}.$$

Check that, with Taylor-Lagrange inequality

$$\begin{aligned}
g_{k+1}(x) - g_k(x) &\leq \max_{t\in[k,k+1]} \left\{ -\frac{2cd^2(x,F)}{(t+1)^2} e^{2c\frac{d^2(x,F)}{t+1}} \right\} \\
&= -2c\left(\frac{d(x,F)}{k+2}\right)^2 g_{k+1}(x).
\end{aligned}$$

In the same way, one has

$$\nabla g_{k+1}(x) \leq \frac{4c[d(x,F)+1]}{k+2} e^{2c\frac{[d(x,F)+1]^2}{k+2}}.$$

Since f is supported in F, then $P^k f$ is supported in $\{x \in \Gamma,\, d(x,F) \leq k\}$. As a consequence, we can assume in the previous calculus that $d(x,F) \leq k$ and thus

$$\frac{[d(x,F)+1]^2}{k+2} \leq \frac{d^2(x,F)}{k+2} + 2.$$

Then

$$\frac{|\nabla g_{k+1}(x)|^2}{4\epsilon_{LB} g_{k+1}(x)} \leq \left(\frac{[d(x,F)+1]}{k+2}\right)^2 \frac{4c^2 e^{8c}}{\epsilon_{LB}} g_{k+1}(x).$$

First case: $d(x,F) \geq 1$, then

$$\frac{|\nabla g_{k+1}(x)|^2}{4\epsilon_{LB} g_{k+1}(x)} \leq \left(\frac{d(x,F)}{k+2}\right)^2 \frac{16c^2 e^{8c}}{\epsilon_{LB}} g_{k+1}(x)$$

and by (20),

$$g_{k+1}(x) - g_k(x) + \frac{|\nabla g_{k+1}(x)|^2}{4\epsilon_{LB} g_{k+1}(x)} \leq 0.$$

Second case, $d(x,F) = 0$, then

$$g_{k+1}(x) - g_k(x) + \frac{|\nabla g_{k+1}(x)|^2}{4\epsilon_{LB} g_{k+1}(x)} \leq \frac{1}{(k+2)^2} \frac{16c^2 e^{8c}}{\epsilon_{LB}^2} \leq \frac{2c}{(k+2)^2}$$

In all cases, one has then $P^k f(x) = 0$ or

$$g_{k+1}(x) - g_k(x) + \frac{|\nabla g_{k+1}(x)|^2}{4\epsilon_{LB} g_{k+1}(x)} \leq \frac{2c}{(k+2)^2}.$$

Lemma 2.9 yields

$$\left\| P^{k+1} f e^{c\frac{d^2(.,F)}{k+2}} \right\|_{L^2}^2 - \left\| P^k f e^{c\frac{d^2(.,F)}{k+1}} \right\|_{L^2}^2 \leq \frac{2c}{(k+2)^2} \|P^k f\|_{L^2}^2,$$

and hence, by induction,

$$\left\| P^k f e^{c\frac{d^2(.,F)}{k+1}} \right\|_{L^2}^2 \leq \|f\|_{L^2}^2 + \sum_{l=0}^{k-1} \frac{2c}{(l+2)^2} \|P^l f\|_{L^2}^2 \lesssim \|f\|_{L^2}^2.$$

Consider now a general $f \in L^2(\Gamma)$. Without loss of generality, we can assume that f is nonnegative. Let $(\Gamma_i)_{i\in\mathbb{N}}$ an increasing sequence of finite subsets of Γ such that $\bigcup_{i=0}^\infty \Gamma_i = \Gamma$. Let $f_i = f \mathbb{1}_{\Gamma_i}$. One has then for any $x \in \Gamma$ and $k \in \mathbb{N}$,

$$f_i \uparrow f \qquad \text{and } P^k f_i \uparrow P^k f.$$

By the monotone convergence theorem, we obtain,

$$\left\| P^k f_i e^{c\frac{d^2(.,F)}{k+1}} \right\|_{L^2}^2 \uparrow \left\| P^k f e^{c\frac{d^2(.,F)}{k+1}} \right\|_{L^2}^2$$

so that

$$\begin{aligned}\left\| P^k f e^{c\frac{d^2(.,F)}{k+1}} \right\|_{L^2}^2 &= \lim_{i\to\infty} \left\| P^k f_i e^{c\frac{d^2(.,F)}{k+1}} \right\|_{L^2}^2 \\ &\lesssim \sup_{i\in\mathbb{N}} \|f_i\|_{L^2}^2 \\ &= \|f\|_{L^2}^2.\end{aligned}$$

□

Proposition 2.10. *Let* (Γ,μ) *satisfying* (LB) *(note that* (DV) *is not assumed here). Let* $c > 0$ *as in Proposition 2.8. There exists* $C > 0$ *such that for all subsets* $F \subset \Gamma$ *and all functions* f *supported in* F*, one has*

$$\left\| \nabla P^k f e^{\frac{c}{2}\frac{d^2(.,F)}{k+1}} \right\|_{L^2} \leq C \frac{\|f\|_{L^2}}{\sqrt{k+1}}.$$

Proof: The proof of this proposition is very similar to the one of Lemma 7 in [24]. We define

$$I = I_k(f) := \left\| \nabla P^k f e^{\frac{c}{2}\frac{d^2(.,F)}{k+1}} \right\|_{L^2}^2 .$$

One has then

$$\begin{aligned}
I &= \sum_{x,y\in\Gamma} p(x,y)|P^k(x) - P^k(y)|^2 e^{c\frac{d^2(x,F)}{k+1}} m(x)m(y) \\
&= \sum_{x,y\in\Gamma} p(x,y)[P^k f(x) - P^k f(y)] P^k f(x) e^{c\frac{d^2(x,F)}{k+1}} m(x)m(y) \\
&\quad - \sum_{x,y\in\Gamma} p(x,y)[P^k f(x) - P^k f(y)] P^k f(y) e^{c\frac{d^2(x,F)}{k+1}} m(x)m(y) \\
&= 2\sum_{x,y\in\Gamma} p(x,y)[P^k f(x) - P^k f(y)] P^k f(x) e^{c\frac{d^2(x,F)}{k+1}} m(x)m(y) \\
&\quad + \sum_{x,y\in\Gamma} p(x,y)[P^k f(x) - P^k f(y)] P^k f(x) \left[e^{c\frac{d^2(y,F)}{k+1}} - e^{c\frac{d^2(x,F)}{k+1}} \right] m(x)m(y) \\
&:= 2I_1 + I_2.
\end{aligned}$$

We first estimate I_1. One has

$$\begin{aligned}
I_1 &= \sum_{x\in\Gamma} P^k f(x) e^{c\frac{d^2(x,F)}{k+1}} m(x) \sum_{y\in\Gamma} p(x,y)[P^k f(x) - P^k f(y)] m(y) \\
&= \sum_{x\in\Gamma} (I-P)P^k f(x) P^k f(x) e^{c\frac{d^2(x,F)}{k+1}} m(x).
\end{aligned}$$

Consequently, with the analyticity of P and Proposition 2.8, we get

$$\begin{aligned}
I_1 &\leq \|(I-P)P^k f\|_{L^2} \left\| P^k f e^{c\frac{d^2(.,F)}{k+1}} \right\|_{L^2} \\
&\lesssim \frac{1}{k+1}\|f\|_{L^2}^2.
\end{aligned} \tag{21}$$

We now turn to the estimate of I_2. One has, since $d(x,y) \leq 1$ (otherwise $p(x,y) = 0$),

$$\begin{aligned}
\left| e^{c\frac{d^2(y,F)}{k+1}} - e^{c\frac{d^2(x,F)}{k+1}} \right| &\leq 2\frac{[d(x,F)+1]}{k+1} e^{c\frac{d^2(x,F)}{k+1}} \\
&\lesssim \frac{1}{\sqrt{k+1}} e^{\frac{3c}{2}\frac{d^2(x,F)}{k+1}}.
\end{aligned}$$

Since f is supported in F, $P^k f$ is supported in $\{x \in \Gamma, d(x,F) \leq k\}$. Consequently, we can assume that $d(x,F) \leq k+1$ so that

$$\frac{[d(x,F)+1]^2}{k+1} \leq \frac{d^2(x,F)}{k+1} + 2.$$

Therefore, the term I_2 can be estimated by

$$\begin{aligned}
|I_2| &\lesssim \frac{1}{\sqrt{k+1}} \sum_{x,y\in\Gamma} |P^k f(x) - P^k f(y)||P^k f(x)| e^{\frac{3c}{2}\frac{d^2(x,F)}{k+1}} m(x)m(y) \\
&\lesssim \frac{1}{\sqrt{k+1}} \left(\sum_{x,y\in\Gamma} |P^k f(x) - P^k f(y)|^2 e^{c\frac{d^2(x,F)}{k+1}} m(x)m(y) \right)^{\frac{1}{2}} \left(\sum_{x,y\in\Gamma} |P^k f(x)|^2 e^{2c\frac{d^2(x,F)}{k+1}} m(x)m(y) \right)^{\frac{1}{2}} \\
&= \frac{1}{\sqrt{k+1}} \sqrt{I} \left\| P^k f e^{c\frac{d^2(.,F)}{k+1}} \right\|_{L^2} \\
&\lesssim \sqrt{\frac{I}{k+1}} \|f\|_{L^2},
\end{aligned} \tag{22}$$

where we used again Proposition 2.8 for the last line.

The estimates (21) and (22) yield

$$I \lesssim \frac{1}{k+1}\|f\|_{L^2}^2 + \sqrt{\frac{I}{k+1}}\|f\|_{L^2},$$

that is

$$I \lesssim \frac{1}{k+1}\|f\|_{L^2}^2,$$

which is the desired conclusion. □

Corollary 2.11. *Let* (Γ,μ) *satisfying* (LB) *(note that* (DV) *is not assumed here). Let* $M\in\mathbb{N}$*. The following families of operators satisfy the Davies-Gaffney estimates*

(i) $s^{M+\frac12}\nabla\Delta^M P^s$,

(ii) $s^{M+\frac12}\nabla\Delta^M (I+s\Delta)^{-M-\frac12}$.

Proof: According to Propositions 2.5 and 2.6, it is enough to check that $\sqrt{s}\nabla P^s$ and $\sqrt{s}\nabla(I+s\Delta)^{-\frac12}$ satisfy Davies-Gaffney estimates.

Indeed, Proposition 2.10 yields, if $E,F\subset\Gamma$, f supported in F and $c>0$ satisfy (20)

$$\begin{aligned}
\|\sqrt{s}\nabla P^s f\|_{L^2(E)} e^{\frac{c}{2}\frac{d(E,F)}{s+1}} &\leq \sqrt{s}\left\|\nabla P^s f e^{\frac{c}{2}\frac{d^2(.,F)}{k+1}}\right\|_{L^2}\\
&\lesssim \frac{\sqrt{s}}{\sqrt{s+1}} \leq 1.
\end{aligned}$$

It suffices now to check that $\sqrt{s}\nabla(I+s\Delta)^{-\frac12}$ satisfies Davies-Gaffney estimates. First notice that

$$\begin{aligned}
\|\sqrt{s}\nabla(I+s\Delta)^{-\frac12} f\|_{L^2} &= \|(s\Delta)^{\frac12}(I+s\Delta)^{-\frac12} f\|_{L^2}\\
&= \left\|(I-(I+s\Delta)^{-1})^{\frac12} f\right\|_{L^2}\\
&\leq \|f\|_{L^2}.
\end{aligned}$$

Then the family of operators $\sqrt{s}\nabla(I+s\Delta)^{-\frac12}$ is L^2-uniformly bounded. Hence, we can suppose without loss of generality that $d(E,F)^2\geq 1+s$. Write,

$$\begin{aligned}
(I+s\Delta)^{-\frac12} f &= \frac{1}{\sqrt{1+s}}\left(I-\frac{s}{1+s}P\right)^{-\frac12} f\\
&= \frac{1}{\sqrt{1+s}}\sum_{k=0}^{\infty} a_k\left(\frac{s}{1+s}\right)^k P^k f
\end{aligned}$$

where $\sum a_k z^k$ is the Taylor serie of the function $(1-z)^{-\frac12}$ and the convergence holds in $L^2(\Gamma)$. Note that $a_k\simeq\sqrt{k+1}$ (see for example [17], Lemma B.1) and

$$\begin{aligned}
\|\sqrt{s}\nabla(I+s\Delta)^{-\frac12} f\|_{L^2(E)} &\lesssim \frac{\sqrt{s}}{\sqrt{1+s}}\sum_{k=0}^{\infty}\frac{1}{\sqrt{1+k}}\left(\frac{s}{1+s}\right)^k\|\nabla P^k f\|_{L^2(E)}\\
&\lesssim \|f\|_{L^2}\sum_{k=0}^{\infty}\frac{1}{1+k}\left(\frac{s}{1+s}\right)^k e^{-c\frac{d(E,F)^2}{1+k}}\\
&\lesssim \|f\|_{L^2}\frac{1}{d(E,F)^2}\sum_{k=0}^{\infty}\left(\frac{s}{1+s}\right)^k e^{-c\frac{d(E,F)^2}{1+k}}\\
&\lesssim \|f\|_{L^2}\frac{1}{d(E,F)^2}\left[\sum_{k=0}^{s} e^{-c\left[\frac{d(E,F)^2}{1+k}+\frac{k}{1+s}\right]}\right.\\
&\qquad\left.+\sum_{k=s+1}^{\infty}\left(\frac{1+s}{1+k}\right)^2 e^{-c\left[\frac{d(E,F)^2}{1+k}+\frac{k}{1+s}\right]}\right]
\end{aligned}$$

where we used (i) for the second estimate and Lemma B.1 for the last one.

Arguing as in the proof of Proposition 2.6, we find

$$\begin{aligned}\|\sqrt{s}\nabla(I+s\Delta)^{-\frac{1}{2}}\|_{L^2(E)} &\lesssim \|f\|_{L^2}\frac{1+s}{d(E,F)^2}e^{-c\frac{d(E,F)}{\sqrt{1+s}}} \\ &\lesssim \|f\|_{L^2},\end{aligned}$$

since we assumed that $d(E,F)^2 \geq 1+s$. □

Corollary 2.12. *Let $M \in \mathbb{N}$. Then if $a = s^{M+\frac{1}{2}}d\Delta^M(I+s\Delta)^{-M-\frac{1}{2}}b$ is a $(BZ_2, M+\frac{1}{2}, \epsilon)$-molecule associated with the ball B, then*

$$\|a\|_{L^1(T_\Gamma)} \lesssim 1 \qquad \text{and} \qquad \|a\|_{L^2(T_{C_j(B)})} \lesssim \frac{2^{-j\epsilon}}{V(2^jB)^{\frac{1}{2}}} \quad \forall j \in \mathbb{N}^*.$$

Proof: First, notice that

$$\|a\|_{L^1(T_\Gamma)} \leq \sum_{j\geq 1} V(2^{j+1}B)^{\frac{1}{2}}\|x \mapsto \|a(x,.)\|_{T_x}\|_{L^2(C_j(B))}.$$

Then it remains to check the last claim, that is

$$\|a\|_{L^2(T_{C_j(B)})} := \|x \mapsto \|a(x,.)\|_{T_x}\|_{C_j(B)} \lesssim \frac{2^{-j\epsilon}}{V(2^jB)^{\frac{1}{2}}}.$$

Since $a = s^{M+\frac{1}{2}}d\Delta^M(I+s\Delta)^{-M-\frac{1}{2}}b$, then

$$x \mapsto \|a(x,.)\|_{T_x} = s^{M+\frac{1}{2}}\nabla\Delta^M(I+s\Delta)^{-M-\frac{1}{2}}b(x).$$

We conclude as in Proposition 2.7, using the Davies-Gaffney estimates provided by Corollary 2.11. □

2.4 Off diagonal decay for Littlewood-Paley functionals

Lemma 2.13. *Let $M > 0$ and $\alpha \in [0,1]$. Define $\mathcal{A} = \{(A_l^{d,u})_{l\in\mathbb{N}^*},\ d \in \mathbb{R}_+,\ u \in \mathbb{N}\}$, where, for all $l \geq 1$,*

$$A_l^{d,u} = l^\alpha \frac{\exp\left(-\frac{d}{l+u}\right)}{(l+u)^{1+M}}.$$

Then there exists $C = C_{M,\alpha}$ such that

$$\left(\sum_{l\in\mathbb{N}^*}\frac{1}{l}a_l^2\right)^{\frac{1}{2}} \leq C\sum_{l\in\mathbb{N}^*}\frac{1}{l}a_l \qquad \forall (a_l)_l \in \mathcal{A}.$$

Proof: The proof is similar to Proposition C.2 in [17]. □

Lemma 2.14. *Let $M \in \mathbb{N}^*$ and $\beta > 0$. Then there exists $C_{M,\beta}$ such that for all sets $E, F \subset \Gamma$, all f supported in F, all $s \in \mathbb{N}$ and all M-tuples $(s_1,\dots,s_M) \in [\![s,2s]\!]^M$, one has*

$$\|L_\beta(I-P^{s_1})\dots(I-P^{s_M})f\|_{L^2(E)} \leq C_{M,\beta}\left(1+\frac{d(E,F)^2}{s}\right)^{-M}\|f\|_{L^2}.$$

Proof: The proof follows the ideas of [17] Lemma 3.3 (or [4] Lemma 3.2 if $\beta = 1$). First, since L_β and $(I-P^{s_1})\dots(I-P^{s_M})$ are L^2-bounded (uniformly in s) and without loss of generality, we can assume that $s \leq d(E,F)^2$.

Denote by η the only integer such that $\eta+1 \geq \beta+M > \eta \geq 0$. Notice that $M-\eta \leq 1-\beta < 1$ and thus $M-\eta \leq 0$.

We use the following fact, which is an immediate consequence of Proposition 2.1

$$\Delta^{\beta+M}f = (I-P)^{\beta+M}f = \sum_{k\geq 0} a_kP^k(I-P)^{\eta+1}f \qquad \forall f \in L^2(\Gamma)$$

where $\sum a_k z^k$ is the Taylor serie of the function $(1-z)^{\beta+M-\eta-1}$ Notice that if $\beta+M$ is an integer, then $a_k = \delta_0(k)$. By the use of the generalized Minkowski inequality, we get

$$\begin{aligned}
&\left\|L_\beta(I-P^s)^M f\right\|_{L^2(E)} \\
&\quad \leq \sum_{k\geq 0} a_k \left(\sum_{l\geq 1} l^{2\beta-1} \sum_{x\in E} \frac{m(x)}{V(x,\sqrt{l})} \sum_{y\in B(x,\sqrt{l})} m(y)|\Delta^{1+\eta-M}(I-P^{s_1})\dots(I-P^{s_M})P^{k+l-1}f(y)|^2\right)^{\frac{1}{2}} \\
&\quad \lesssim s^M \sup_{t\in[\![0,2Ms]\!]} \sum_{k\geq 0} a_k \left(\sum_{l\geq 1} l^{2\beta-1} \sum_{x\in E} \frac{m(x)}{V(x,\sqrt{l})} \sum_{y\in B(x,\sqrt{l})} m(y)|\Delta^{1+\eta}P^{k+l+t-1}f(y)|^2\right)^{\frac{1}{2}} \\
&\quad \lesssim s^M \sup_{t\in[\![0,2Ms]\!]} \sum_{k\geq 0} a_k \left(\sum_{l\geq 1} l^{2\beta-1} \sum_{y\in D_l(E)} m(y)|\Delta^{1+\eta}P^{k+l+t-1}f(y)|^2 \sum_{x\in B(y,\sqrt{l})} \frac{m(x)}{V(x,\sqrt{l})}\right)^{\frac{1}{2}} \\
&\quad \lesssim s^M \sup_{t\in[\![0,2Ms]\!]} \sum_{k\geq 0} a_k \left(\sum_{l\geq 1} l^{2\beta-1} \|\Delta^{1+\eta}P^{k+l+t-1}f\|^2_{L^2(D_l(E))}\right)^{\frac{1}{2}} \\
&\qquad := s^M \sup_{t\in[\![0,Ms]\!]} \Lambda(t)
\end{aligned}$$

where $D_l(E) = \{y \in \Gamma,\ \mathrm{dist}(y,E) < \sqrt{l}\}$, and where we notice that $\sum_{x\in B(y,\sqrt{l})} \frac{m(x)}{V(x,\sqrt{l})} \lesssim 1$ with the doubling property.

1- Estimate when $\mathbf{l < \frac{d(E,F)^2}{4}}$

The important point here is to notice that $\mathrm{dist}(F, D_l(E)) \geq \frac{1}{2} d(E,F) \gtrsim d(E,F)$. Then, using Davies-Gaffney estimates (Proposition 2.6, (iii)) , we may obtain

$$\begin{aligned}
\|\Delta^{1+\eta}P^{k+l+t-1}f\|_{L^2(D_l(E))} &\lesssim \frac{\exp\left(-c\frac{d(E,F)^2}{l+k+t}\right)}{(l+k+t)^{(1+\eta)}}\|f\|_{L^2} \\
&\leq l^{M-\eta}\frac{\exp\left(-c\frac{d(E,F)^2}{l+k+t}\right)}{(l+k+t)^{1+M}}\|f\|_{L^2}
\end{aligned} \tag{23}$$

since $M-\eta \leq 0$.

2- Estimate when $\mathbf{l \geq \frac{d(E,F)^2}{4}}$

We use the analyticity of P to obtain,

$$\begin{aligned}
\|\Delta^{1+\eta}P^{k+l+t-1}f\|_{L^2(D_l(E))} &\leq \|(I-P)^{1+\eta}P^{k+l+t-1}f\|_{L^2(\Gamma)} \\
&\lesssim \frac{1}{(k+l+t)^{1+\eta}}\|f\|_{L^2} \\
&\lesssim l^{M-\eta}\frac{1}{(k+l+t)^{1+M}}\|f\|_{L^2} \\
&\lesssim l^{M-\eta}\frac{\exp\left(-c\frac{d(E,F)^2}{l+k+t}\right)}{(l+k+t)^{1+M}}\|f\|_{L^2}
\end{aligned} \tag{24}$$

where the third line is due to $M-\eta \leq 0$ and the last one holds because $l+k \gtrsim d(E,F)^2$.

3- Conclusion

The first two steps imply the following estimate on $\Lambda(t)$:

$$\Lambda(t) \lesssim \|f\|_{L^2(F)} \sum_{k\geq 0} a_k \left(\sum_{l\geq 1} \frac{1}{l} \left| l^{(\beta+M-\eta)} \frac{\exp\left(-c\frac{d(E,F)^2}{l+k+t}\right)}{(k+l+t)^{1+M}} \right|^2 \right)^{\frac{1}{2}}$$
$$\lesssim \|f\|_{L^2(F)} \sum_{k\geq 0} a_k \sum_{l\geq 1} \frac{1}{l} l^{(\beta+M-\eta)} \frac{\exp\left(-c\frac{d(E,F)^2}{l+k+t}\right)}{(k+l+t)^{1+M}}$$

where we used Lemma 2.13 for the last line (indeed, $\beta+M-\eta \in (0,1]$). Check that Thus since

$$\sum_{k=0}^{m-1} a_k (m-k)^{\beta+M-\eta-1} \lesssim 1.$$

Indeed, when $\beta+M-\eta = 1$, the result is obvious. Otherwise, it is a consequence of the fact that $a_k \simeq k^{\eta-M-\beta}$ (see Lemma B.1 in [17]). Hence, one has

$$\begin{aligned}\Lambda(t) &\lesssim \|f\|_{L^2(F)} \sum_{m\geq 1} \frac{\exp\left(-c\frac{d(E,F)^2}{m+t}\right)}{(m+t)^{1+M}} \\ &= d(E,F)^{-2(1+M)} \|f\|_{L^2(F)} \sum_{m\geq 1} \frac{d(E,F)^{2(1+M)}}{(m+t)^{1+M}} \exp\left(-c\frac{d(E,F)^2}{m+t}\right) \\ &\lesssim d(E,F)^{-2(1+M)} \|f\|_{L^2(F)} \left[\sum_{m=1}^{d(E,F)^2} 1 + \sum_{m>d(E,F)^2} \frac{d(E,F)^{2(1+M)}}{(m+t)^{1+M}} \right] \\ &\lesssim d(E,F)^{-2M} \|f\|_{L^2(F)}.\end{aligned}$$

As a consequence,

$$\left\| L_\beta (I-P^s)^M f \right\|_{L^2(E)} \lesssim \left(\frac{d(E,F)^2}{s} \right)^{-M} \|f\|_{L^2(F)}$$

which is the desired conclusion.

□

Lemma 2.15. *Let $M \in \mathbb{R}_+^*$ and $\beta > 0$ such that either $M \in \mathbb{N}$ or $\beta \geq 1$. Define the Littlewood-Paley functional G_β on $L^2(\Gamma)$ by*

$$G_\beta f(x) = \left(\sum_{l\leq 1} l^{2\beta-1} |\Delta^\beta P^{l-1} f(x)|^2 \right)^{\frac{1}{2}} \qquad \forall x \in \Gamma.$$

Then there exists $C_M > 0$ such that for all sets $E, F \subset \Gamma$, all functions f supported in F and all $s \in \mathbb{N}$, one has

$$\left\| G_\beta (s\Delta)^M f \right\|_{L^2(E)} \leq C_M \left(\frac{d(E,F)^2}{s} \right)^{-M} \|f\|_{L^2}.$$

Proof: The proof is similar to the one of Lemma 2.14. Notice that $\sup_{t\in [\![0,Ms]\!]} \Lambda(t)$ is replaced by $\Lambda := \Lambda(0)$. Then the end of the calculus is the same provided that $\eta - M \geq 0$, which is the case under our assumption on M and β. See also [17, Lemma 3.3]. □

Lemma 2.16. *Let $M \in \mathbb{N}$. Then there exists $C_M > 0$ such that for all sets $E, F \subset \Gamma$, all f supported in F and all $s \in \mathbb{N}$, one has*

$$\left\| L_{\frac{1}{2}} (I - (I+s\Delta)^{-1})^{M+\frac{1}{2}} f \right\|_{L^2(E)} \leq C_M \left(1 + \frac{d(E,F)^2}{s} \right)^{-M-\frac{1}{2}} \|f\|_{L^2}.$$

Proof: Since $L_{\frac{1}{2}}$ and $(I-(I+s\Delta)^{-1})^{M+\frac{1}{2}}$ are L^2-bounded (uniformly in s) and without loss of generality, we can assume that $s \leq d(E,F)^2$.

We use the following computation,

$$\begin{aligned}(I+s\Delta)^{-M-\frac{1}{2}}f &= ((1+s)I-sP)^{-M-\frac{1}{2}} = (1+s)^{-M-\frac{1}{2}}\left(I-\frac{s}{1+s}P\right)^{-M-\frac{1}{2}}f \\ &= (1+s)^{-M-\frac{1}{2}}\sum_{k\geq 0} a_k\left(\frac{s}{1+s}\right)^k P^k f\end{aligned} \tag{25}$$

where $\sum a_k z^k$ is the Taylor series of the function $(1-z)^{-M-\frac{1}{2}}$ and the convergence holds in $L^2(\Gamma)$.

By the use of the generalized Minkowski inequality, we get

$$\begin{aligned}&\left\|L_{\frac{1}{2}}(I-(I+s\Delta)^{-1})^{M+\frac{1}{2}}f\right\|_{L^2(E)} \\ &\leq \frac{s^{M+\frac{1}{2}}}{(1+s)^{M+\frac{1}{2}}}\sum_{k\geq 0}a_k\left(\frac{s}{1+s}\right)^k\left(\sum_{l\geq 1}\sum_{x\in E}\frac{m(x)}{V(x,\sqrt{l})}\sum_{y\in B(x,\sqrt{l})}m(y)|\Delta^{1+M}P^{k+l-1}f(y)|^2\right)^{\frac{1}{2}} \\ &\leq \frac{s^{M+\frac{1}{2}}}{(1+s)^{M+\frac{1}{2}}}\sum_{k\geq 0}a_k\left(\frac{s}{1+s}\right)^k\left(\sum_{l\geq 1}\sum_{y\in D_l(E)}m(y)|\Delta^{1+M}P^{k+l-1}f(y)|^2\sum_{x\in B(y,\sqrt{l})}\frac{m(x)}{V(x,\sqrt{l})}\right)^{\frac{1}{2}} \\ &\lesssim \frac{s^{M+\frac{1}{2}}}{(1+s)^{M+\frac{1}{2}}}\sum_{k\geq 0}a_k\left(\frac{s}{1+s}\right)^k\left(\sum_{l\geq 1}\|\Delta^{1+M}P^{k+l-1}f\|^2_{L^2(D_l(E))}\right)^{\frac{1}{2}}.\end{aligned}$$

When $l < \frac{d^2(E,F)}{4}$, notice that $d(F,D_l(E)) \gtrsim d(E,F)$ so that

$$\|\Delta^{1+M}P^{k+l-1}f\|_{L^2(D_l(E))} \lesssim \frac{\exp(-c\frac{d^2(E,F)}{l+k})}{(l+k)^{M+1}}\|f\|_{L^2}. \tag{26}$$

Moreover, when $l \geq \frac{d^2(E,F)}{4}$, one has

$$\begin{aligned}\|\Delta^{1+M}P^{k+l-1}f\|_{L^2(D_l(E))} &\leq \|\Delta^{1+M}P^{k+l-1}f\|_{L^2} \\ &\lesssim \frac{1}{(l+k)^{M+1}}\|f\|_{L^2} \\ &\lesssim \frac{\exp(-c\frac{d^2(E,F)}{l+k})}{(l+k)^{M+1}}\|f\|_{L^2}.\end{aligned} \tag{27}$$

As a consequence

$$\begin{aligned}\left\|L_{\frac{1}{2}}(I-(I+s\Delta)^{-1})^{M+\frac{1}{2}}f\right\|_{L^2(E)} &\lesssim \frac{s^{M+\frac{1}{2}}}{(1+s)^{M+\frac{1}{2}}}\|f\|_{L^2}\sum_{k\geq 0}a_k\left(\frac{s}{1+s}\right)^k\left(\sum_{l\geq 1}\frac{\exp(-c\frac{d^2(E,F)}{l+k})}{(l+k)^{2(M+1)}}\right)^{\frac{1}{2}} \\ &\lesssim \frac{s^{M+\frac{1}{2}}}{(1+s)^{M+\frac{1}{2}}}\|f\|_{L^2}\sum_{k\geq 0}a_k\left(\frac{s}{1+s}\right)^k\left(\sum_{n\geq 1}\frac{\exp(-c\frac{d^2(E,F)}{n})}{n^{2(M+1)}}\right)^{\frac{1}{2}} \\ &\lesssim \frac{s^{M+\frac{1}{2}}}{d(E,F)^{2M+1}}\frac{1}{(1+s)^{M+\frac{1}{2}}}\|f\|_{L^2}\sum_{k\geq 0}a_k\left(\frac{s}{1+s}\right)^k \\ &= \left(\frac{s}{d(E,F)^2}\right)^{M+\frac{1}{2}}(1+s(1-1))^{-M-\frac{1}{2}}\|f\|_{L^2} \\ &= \left(\frac{s}{d(E,F)^2}\right)^{M+\frac{1}{2}}\|f\|_{L^2}.\end{aligned}$$

$\square$

Let us now recall a result that can be found in [17], Theorem 1.4.

Proposition 2.17. *Assume that* (Γ, μ) *satisfy* (UE). *Let* $K > 0$ *and* $j \in \mathbb{N}$. *There exist* $C, c > 0$ *such that for all sets* $E, F \in \Gamma$ *and all* $x_0 \in \Gamma$ *all* $l \in \mathbb{N}^*$ *satisfying*

$$\sup_{y \in F} d(x_0, y) \leq K d(E, F) \tag{28}$$

or

$$\sup_{y \in F} d(x_0, y) \leq K\sqrt{l} \tag{29}$$

and all functions f *supported in* F, *there holds*

$$\|\Delta^j P^{l-1} f\|_{L^2(E)} \leq \frac{C}{l^j} \frac{1}{V(x_0, \sqrt{l})^{\frac{1}{2}}} e^{-c\frac{d(E,F)^2}{l}} \|f\|_{L^1(F)}$$

and

$$\|\nabla \Delta^j P^{l-1} f\|_{L^2(E)} \leq \frac{C}{l^{j+\frac{1}{2}}} \frac{1}{V(x_0, \sqrt{l})^{\frac{1}{2}}} e^{-c\frac{d(E,F)^2}{l}} \|f\|_{L^1(F)}.$$

Lemma 2.18. *Assume that* (Γ, μ) *satisfy* (UE). *For all* $M \in \mathbb{N}^*$ *and all* $\beta > 0$, *there exists* $C_M > 0$ *such that for all disjoint sets* $E, F \in \Gamma$ *and all* x_0 *satisfying* (28), *all* f *supported in* F *and all* $s \in \mathbb{N}^*$, *one has*

$$\left\|L_\beta (I - P^s)^M f\right\|_{L^2(E)} \leq \frac{C_M}{V(x_0, d(E,F))^{\frac{1}{2}}} \left(\frac{d(E,F)^2}{s}\right)^{-M} \|f\|_{L^1}.$$

Proof: The proof of this Lemma is similar to the one of Lemma 2.14 and we only indicate the main changes.

When $l < \frac{d(E,F)^2}{4}$, replace first (23) by

$$\begin{aligned} \|\Delta^{1+\eta} P^{k+l+t-1} f\|_{L^2(D_l(E))} &\lesssim \frac{1}{V(x_0, \sqrt{k+l+t})^{\frac{1}{2}}} \frac{\exp\left(-c\frac{d(E,F)^2}{l+k+t}\right)}{(l+k+t)^{(1+\eta)}} \|f\|_{L^1} \\ &\lesssim \frac{l^{M-\eta}}{V(x_0, d(E,F))^{\frac{1}{2}}} \frac{\exp\left(-c\frac{d(E,F)^2}{l+k+t}\right)}{(l+k+t)^{1+M}} \|f\|_{L^1} \end{aligned} \tag{30}$$

where the second line holds because $M - \eta \leq 0$ and the first one holds by Proposition 2.17. Indeed, there exists $K > 0$ such that

$$\sup_{y \in F} d(x_0, y) \leq K d(E, F).$$

Thus, x_0, $D_l(E)$ and F satisfy (28) with constant $4K$.

When $l \geq \frac{d(E,F)^2}{4}$, replace also (24) by

$$\begin{aligned} \|\Delta^{1+\eta} P^{k+l+t-1} f\|_{L^2(D_l(E))} &\lesssim \frac{1}{V(x_0, \sqrt{k+l+t})^{\frac{1}{2}}} \frac{1}{(k+l+t)^{1+\eta}} \|f\|_{L^1} \\ &\lesssim \frac{l^{M-\eta}}{V(x_0, d(E,F))^{\frac{1}{2}}} \frac{\exp\left(-c\frac{d(E,F)^2}{l+k+t}\right)}{(k+l+t)^{1+M}} \|f\|_{L^1} \end{aligned} \tag{31}$$

where the first line follows from Proposition 2.17, since x_0, F and $k+l+t$ satisfy (29), and the second line to the facts that $k + l \gtrsim d(E,F)^2$ and $M - \eta \leq 0$. $\square$

Lemma 2.19. *Assume that* (Γ, μ) *satisfy* (UE). *For all* $M > 0$, *there exists* C_M *such that for all sets* $E, F \in \Gamma$ *and all* x_0 *satisfying* (28), *all* f *supported in* F *and all* $s \in \mathbb{N}^*$, *one has*

$$\left\|L_{\frac{1}{2}} (I - (I + s\Delta)^{-1})^{M+\frac{1}{2}} f\right\|_{L^2(E)} \leq \frac{C_M}{V(x_0, d(E,F))^{\frac{1}{2}}} \left(\frac{d(E,F)^2}{s}\right)^{-M-\frac{1}{2}} \|f\|_{L^1}.$$

Proof: The proof of this Lemma is similar to the one of Lemma 2.16 and we only indicate the main changes.

When $l < \frac{d(E,F)^2}{4}$, replace (26) by

$$\begin{aligned}\|\Delta^{1+M} P^{k+l-1} f\|_{L^2(D_l(E))} &\lesssim \frac{1}{V(x_0, \sqrt{k+l})^{\frac{1}{2}}} \frac{\exp\left(-c\frac{d(E,F)^2}{l+k}\right)}{(l+k)^{(1+M)}} \|f\|_{L^1} \\ &\lesssim \frac{1}{V(x_0, d(E,F))^{\frac{1}{2}}} \frac{\exp\left(-c\frac{d(E,F)^2}{l+k}\right)}{(l+k)^{1+M}} \|f\|_{L^1}\end{aligned} \tag{32}$$

where the first line holds due to Lemma 2.17 since x_0, $D_l(E)$ and F satisfy (28).

When $l \geq \frac{d(E,F)^2}{4}$, replace also (27) by

$$\begin{aligned}\|\Delta^{1+M} P^{k+l-1} f\|_{L^2(D_l(E))} &\lesssim \frac{1}{V(x_0, \sqrt{k+l})^{\frac{1}{2}}} \frac{1}{(k+l)^{1+M}} \|f\|_{L^1} \\ &\lesssim \frac{1}{V(x_0, d(E,F))^{\frac{1}{2}}} \frac{\exp\left(-c\frac{d(E,F)^2}{l+k}\right)}{(k+l)^{1+M}} \|f\|_{L^1}\end{aligned} \tag{33}$$

where the second line follows from Lemma 2.17, since x_0, F and $k+l+t$ satisfy (29), and the third line to the fact that $k+l \gtrsim d(E,F)^2$. □

3 BMO spaces

3.1 Dense sets in Hardy spaces

Lemma 3.1. *Let $M \in \mathbb{N}$ and $\kappa \in \{1,2\}$.*

For all $\epsilon \in (0,+\infty)$, we have the following inclusion

$$\mathcal{M}_0^{M,\epsilon}(\Gamma) \hookrightarrow H^1_{BZ\kappa,M,\infty}(\Gamma)$$

and for all $\phi \in \mathcal{M}_0^{M,\epsilon}(\Gamma)$,

$$\|\phi\|_{H^1_{BZ\kappa,M,\infty}} \leq C_{M,\epsilon} \|\phi\|_{\mathcal{M}_0^{M,\epsilon}}.$$

Proof: Let ϕ in $\mathcal{M}_0^{M,\epsilon}(\Gamma)$. Then there exists $\varphi \in L^2(\Gamma)$ such that $\phi = \Delta^M \varphi$ and for all $j \geq 1$,

$$\|\varphi\|_{L^2(C_j(B_0))} 2^{j\epsilon} \lesssim \|\phi\|_{\mathcal{M}_0^{M,\epsilon}}.$$

Observe that

$$\varphi(x) = \sum_{y\in\Gamma} a_y \frac{\mathbb{1}_{\{y\}}(x)}{m(y)} \qquad \forall x \in \Gamma \tag{34}$$

where $a_y = \varphi(y)m(y)$. In order to prove that $\phi \in H^1_{BZ\kappa,M,\infty}$, it suffices to prove

(i) for every $y \in \Gamma$, $\Delta^M \frac{\mathbb{1}_{\{y\}}}{m(y)}$ is, up to a harmless multiplicative constant, a (BZ_κ, M)-atom,

(ii) $\sum_{y\in\Gamma} |a_y| \lesssim \|\phi\|_{\mathcal{M}_0^{M,\epsilon}}$,

(iii) $\phi = \sum_{y\in\Gamma} a_y \Delta^M \frac{\mathbb{1}_{\{y\}}}{m(y)}$ where the convergence holds in $L^1(\Gamma)$.

It is easy to check that $\Delta^M \frac{\mathbb{1}_{\{y\}}}{m(y)}$ is a (BZ_1, M)-atom associated with $s = 1$, $(1, \dots, 1)$ and the ball $B(y,1)$. When $\kappa = 2$, notice that

$$\Delta^M \frac{\mathbb{1}_{\{y\}}}{m(y)} = \left(I - (I + (M^2+1)\Delta)^{-1}\right)^M \left(\frac{I + (M^2+1)\Delta}{M^2+1}\right)^M \frac{\mathbb{1}_{\{y\}}}{m(y)}.$$

Moreover, $\left(\frac{I+(M^2+1)\Delta}{M^2+1}\right)^M \frac{1_{\{y\}}}{m(y)}$ is supported in $B(y, M+1)$ and

$$\left\| \left(\frac{I+(M^2+1)\Delta}{M^2+1}\right)^M \frac{\mathbb{1}_{\{y\}}}{m(y)} \right\|_{L^2} \leq \left(\frac{2M^2+3}{M^2+1}\right)^M \left\| \frac{\mathbb{1}_{\{y\}}}{m(y)} \right\|_{L^2}$$
$$\lesssim \frac{1}{m(y)^{\frac{1}{2}}}$$
$$\lesssim \frac{1}{V(y, M+1)^{\frac{1}{2}}}.$$

For point (ii), remark that

$$\begin{aligned}
\sum_{y\in\Gamma} |a_y| &= \sum_{j\geq 1} \sum_{y\in C_j(B_0)} |a_y| \\
&\leq \sum_{j\geq 1} \left(\sum_{y\in C_j(B_0)} \frac{|a_y|^2}{m(y)} \right)^{\frac{1}{2}} (m(C_j(B_0)))^{\frac{1}{2}} \\
&\lesssim \sum_{j\geq 1} V(2^j B_0)^{\frac{1}{2}} \left(\sum_{y\in C_j(B_0)} |\varphi(y)|^2 m(y) \right)^{\frac{1}{2}} \\
&= \sum_{j\geq 1} V(2^j B_0)^{\frac{1}{2}} \|\varphi\|_{L^2(C_j(B_0))} \\
&\leq \sum_{j\geq 1} 2^{-j\epsilon} \|\phi\|_{\mathcal{M}_0^{M,\epsilon}} \\
&\lesssim \|\phi\|_{\mathcal{M}_0^{M,\epsilon}}.
\end{aligned}$$

For point (iii), notice that (ii) implies the L^1-convergence in (34). The result is then a consequence of the L^1-boundedness of Δ. $\square$

Lemma 3.2. *Let $M \in \mathbb{N}^*$ and let $B \subset \Gamma$ be a ball. For all $s \in \mathbb{N}^*$, define A_s as either $(I - P^{s_1}) \dots (I - P^{s_M})$ with $(s_1, \dots, s_M) \in [\![1, 2s]\!]^M$, or $(I - (I + s\Delta)^{-1})^M$. If $\varphi \in L^2(B)$ then, for all $s \in \mathbb{N}^*$, $\epsilon > 0$ and $M \in \mathbb{N}^*$, $A_s\varphi \in \mathcal{M}_0^{M,\epsilon}(\Gamma)$.*

As a consequence, if $f \in \mathcal{E}_M$ for some $M \in \mathbb{N}$, then for all $s \in \mathbb{N}$ we can define $A_s f$ as a linear form on finitely supported functions and

$$A_s f \in L^2_{loc}(\Gamma)$$

Remark 3.3. *In the case of graphs, a distribution g is in $L^2_{loc}(\Gamma)$ means that we can write $g(x)$ for all $x \in \Gamma$, that is g is a function. On the contrary, notice that each function on Γ belongs to $L^2_{loc}(\Gamma)$ and we use then the notation $L^2_{loc}(\Gamma)$ only by analogy to the case of continuous spaces.*

Proof: Fix $\epsilon > 0$ and let $\varphi \in L^2(B)$ for some ball B and $k \in \mathbb{N}$ such that $B \subset 2^{k+2}B_0$. The uniform L^2-boundedness of $A_s(s\Delta)^{-M}$ yields

$$\sup_{j\in[\![1,k+1]\!]} 2^{j\epsilon} V(2^j B_0)^{\frac{1}{2}} \left\| A_s \Delta^{-M} \varphi \right\|_{L^2(C_j(B_0))} \lesssim s^M 2^{k\epsilon} V(2^k B_0)^{\frac{1}{2}} \|\varphi\|_{L^2(B)}.$$

Moreover, Proposition 2.6 implies, for $j \geq k+2$

$$\begin{aligned}
2^{j\epsilon} V(2^j B_0)^{\frac{1}{2}} \left\| A_s \Delta^{-M} \varphi \right\|_{L^2(C_j(B_0))} &\lesssim s^M 2^{j\epsilon} V(2^j B_0)^{\frac{1}{2}} e^{-c\frac{2^j}{\sqrt{s}}} \|\varphi\|_{L^2(B)} \\
&\lesssim s^{M+\frac{\epsilon}{2}+\frac{d_0}{4}+1} V(B_0)^{\frac{1}{2}} \|\varphi\|_{L^2(B)}
\end{aligned}$$

where d_0 is given by Proposition 1.5. One concludes that $A_s\varphi \in \mathcal{M}_0^{M,\epsilon}(\Gamma)$ and

$$\|A_s\varphi\|_{\mathcal{M}_0^{M,\epsilon}} \lesssim s^{M+\frac{\epsilon}{2}+\frac{d_0}{4}+1} 2^{k\epsilon} V(2^k B_0)^{\frac{1}{2}} \|\varphi\|_{L^2(B)}. \tag{35}$$

Let us prove the second claim of the lemma. Let ϵ such that $f \in (\mathcal{M}_0^{M,\epsilon}(\Gamma))^*$. For all balls B and all functions φ supported in B, one has

$$\begin{aligned}|\langle A_s f, \varphi\rangle| &:= |\langle f, A_s\varphi\rangle| \\ &\lesssim \|f\|_{(\mathcal{M}_0^{M,\epsilon})^*} \|A_s\varphi\|_{\mathcal{M}_0^{M,\epsilon}} \\ &\lesssim \|f\|_{(\mathcal{M}_0^{M,\epsilon})^*} \|\varphi\|_{L^2(B)},\end{aligned}$$

which proves the lemma since the estimate works for any ball B and any $\varphi \in L^2(B)$. □

Definition 3.4. *Let $\kappa \in \{1,2\}$ and $M \in \mathbb{N}^*$. Define $\mathbb{H}^1_{BZ\kappa,M,\epsilon}(\Gamma)$ as the subset of $H^1_{BZ\kappa,M,\epsilon}(\Gamma)$ made of the functions g that can be written as $g = \sum_{i=0}^N \lambda_i a_i$ where $\lambda_i \in \mathbb{R}$ and a_i is a (BZ_κ, M, ϵ)-molecule and*

$$\sum_{i=0}^N |\lambda_i| \lesssim 2\|g\|_{H^1_{A,\epsilon}}.$$

Lemma 3.5. *For $\kappa \in \{1,2\}$ and $M \in \mathbb{N}^*$, the set $\mathbb{H}^1_{BZ\kappa,M,\epsilon}(\Gamma)$ is dense in $H^1_{BZ\kappa,M,\epsilon}(\Gamma)$.*

Remark 3.6. *This lemma is identical to Lemma 4.5 in [5]. However, we present here a different proof.*

Proof: Let $\kappa \in \{1,2\}$ and $M \in \mathbb{N}^*$.

Let $f \in H^1_{BZ\kappa,M,\epsilon}(\Gamma)$. There exist a numerical sequence $(\lambda_i)_{i\in\mathbb{N}} \in \ell^1(\mathbb{N})$ and a sequence $(a_i)_{i\in\mathbb{N}}$ of (BZ_κ, M, ϵ)-molecules such that $f = \sum \lambda_i a_i$ and

$$\sum_{i\in\mathbb{N}} |\lambda_i| \leq \frac{3}{2}\|f\|_{H^1_{BZ\kappa,M,\epsilon}}.$$

Let $\eta \in \left(0, \frac{1}{4}\right)$. There exists $N \in \mathbb{N}$ such that $\sum_{i>N} |\lambda_i| \leq \eta\|f\|_{H^1_{BZ\kappa,M,\epsilon}}$. We set $g = \sum_{i=0}^{N} \lambda_i a_i$. Then

$$\begin{aligned}\|f-g\|_{H^1_{BZ\kappa,M,\epsilon}} &= \Big\|\sum_{i>N} \lambda_i a_i\Big\|_{H^1_{BZ\kappa,M,\epsilon}} \\ &\leq \sum_{i>N} |\lambda_i| \leq \eta\|f\|_{H^1_{BZ\kappa,M,\epsilon}}\end{aligned}$$

and, therefore $\|f\|_{H^1_{BZ\kappa,M,\epsilon}} \leq \|g\|_{H^1_{BZ\kappa,M,\epsilon}} + \eta\|f\|_{H^1_{BZ\kappa,M,\epsilon}}$, which implies

$$\begin{aligned}\sum_{i=0}^N |\lambda_i| &\leq \frac{3}{2}\|f\|_{H^1_{BZ\kappa,M,\epsilon}} \\ &\leq \frac{3}{2(1-\eta)}\|g\|_{H^1_{BZ\kappa,M,\epsilon}} \\ &\leq 2\|g\|_{H^1_{BZ\kappa,M,\epsilon}}.\end{aligned}$$

□

Lemma 3.7. *Let $\kappa \in \{1,2\}$ and $M \in \mathbb{N}$. Let $0 < \epsilon < \bar\epsilon \leq +\infty$. Then $\mathbb{H}^1_{BZ\kappa,M,\bar\epsilon}(\Gamma) \subset \mathcal{M}_0^{M,\epsilon}$.*

Proof: Since $\mathcal{M}_0^{M,\epsilon}$ is a vector space, it is enough to prove that for each $(BZ_\kappa, M, \bar\epsilon)$-molecule a, one has $a \in \mathcal{M}_0^{M,\epsilon}$.

Notice that the case $\bar\epsilon = \infty$ is proven in Lemma 3.2. Let $\bar\epsilon < +\infty$ and $a = A_s b$ be a $(BZ_\kappa, M, \bar\epsilon)$-molecule associated with $s \in \mathbb{N}^*$ and the ball B of radius $\sqrt{s}$. For all $j \geq 1$, Corollary A.2 provides a covering of $C_j(B)$ with balls of radius $\sqrt{s}$ and with bounded overlapping. We label these balls as $(B_i)_{i\in I_j}$. . Consequently,

$$\begin{aligned}\|a\|_{\mathcal{M}_0^{M,\epsilon}} &\leq \sum_{j\geq 1} \|A_s(b\mathbb{1}_{C_j(B)})\|_{\mathcal{M}_0^{M,\epsilon}} \\ &\leq \sum_{j\geq 1}\sum_{i\in I_j} \|A_s(b\mathbb{1}_{B_i})\|_{\mathcal{M}_0^{M,\epsilon}}.\end{aligned}$$

Moreover, $d(B_i, B_0) \lesssim 2^{j+k}$ where k is such that $B \subset 2^{k+2}B_0$. Thus Lemma 3.2 implies

$$\begin{aligned}
\|a\|_{\mathcal{M}_0^{M,\epsilon}} &\leq C_s \sum_{j\geq 1}\sum_{i\in I_j} 2^{(j+k)\epsilon} V(2^{j+k}B_0)^{\frac{1}{2}} \|b\|_{L^2(B_i)} \\
&\leq C_s \sum_{j\geq 1} 2^{(j+k)\epsilon} V(2^{j+k}B_0)^{\frac{1}{2}} \|b\|_{L^2(\tilde{C}_j(B))} \\
&\leq C_s \sum_{j\geq 1} \frac{2^{(j+k)\epsilon}}{2^{j\bar{\epsilon}}} \left(\frac{V(2^{j+k}B_0)}{V(2^{j-1}B)}\right)^{\frac{1}{2}} \\
&\leq C_s 2^{k(\epsilon+\frac{d}{2})} \sum_{j\geq 1} 2^{j(\epsilon-\bar{\epsilon})} \\
&< +\infty
\end{aligned}$$

where $\tilde{C}_j$ denote $C_j(B) \cup C_{j-1}(B) \cup C_{j+1}(B)$, and where we use the definition of a $(BZ_\kappa, M, \bar{\epsilon})$-molecule for the third line and the fact that $2^{j+k}B_0 \subset 2^{j+k+2}B$. □

3.2 Inclusions between BMO spaces

Lemma 3.8. *There exists $C > 0$ such that for all $s \in \mathbb{N}^*$, all M-tuples $(s_1, \dots, s_M) \in [\![s, 2s]\!]^M$, all balls B of radius $\sqrt{s}$ and all functions $f \in BMO_{BZ2,M}$, one has*

$$\|(I - P^{s_1}) \dots (I - P^{s_M})f\|_{L^2(B)} \leq CV(B)^{\frac{1}{2}} \|f\|_{BMO_{BZ2,M}}.$$

Proof: For $s \in \mathbb{N}^*$, the operator Q_s stands for

$$Q_s := \frac{1}{s}\sum_{k=0}^{s-1} P^k = (I - P^s)(s\Delta)^{-1}.$$

For all $s \in \mathbb{N}^*$, all $s_0 \in [\![s, 2s]\!]$ and all $f \in \mathcal{E}_0$, one has

$$\begin{aligned}
(I - P^{s_0})f &= (I - P^{s_0})(I + s\Delta)(I + s\Delta)^{-1} f \\
&= \left(\frac{1}{s}\sum_{k=0}^{s_0-1} P^k f\right)(I + s\Delta)s\Delta(I + s\Delta)^{-1} f \\
&= \left[\frac{s_0}{s} Q_{s_0} + (I - P^{s_0})\right](I - (I + s\Delta)^{-1}) f
\end{aligned}$$

Recall that all terms make sense and are in $L^2_{loc}(\Gamma)$, according to Lemma 3.2. As a consequence, for $(s_1, \dots, s_M) \in [\![s, 2s]\!]^M$, one has

$$(I - P^{s_1}) \dots (I - P^{s_M})f = \prod_{i=1}^{M}\left[\frac{s_i}{s} Q_{s_i} + (I - P^{s_i})\right](I - (I + s\Delta)^{-1})^M f \tag{36}$$

Since $\frac{s_i}{s} \leq 2$, Proposition 2.6 yields that $\prod_{i=1}^{M}\left[\frac{s_i}{s} Q_{s_i} + (I - P^{s_i})\right]$ satisfies Gaffney-Davies estimates. Hence,

$$\begin{aligned}
\|(I - P^{s_1}) \dots (I - P^{s_M})f\|_{L^2(B)} &\leq \sum_{j\geq 1}\left\|\prod_{i=1}^{M}\left[\frac{s_i}{s} Q_{s_i} + (I - P^{s_i})\right]\left[\mathbb{1}_{C_j(B)}(I - (I + s\Delta)^{-1})^M f\right]\right\|_{L^2(B)} \\
&\lesssim \sum_{j\geq 1} e^{-c4^j} \|(I - (I + s\Delta)^{-1})^M f\|_{L^2(C_j(B))} \\
&\lesssim \sum_{j\geq 1} e^{-c4^j} \|(I - (I + s\Delta)^{-1})^M f\|_{L^2(2^{j+1}B)} \\
&\lesssim \sum_{j\geq 1} e^{-c4^j} V(2^{j+1}B)^{\frac{1}{2}} \|f\|_{BMO_{BZ2,M}} \\
&\lesssim V(B)^{\frac{1}{2}} \|f\|_{BMO_{BZ2,M}}
\end{aligned}$$

where the last line holds thanks to Proposition 1.5. □

Corollary 3.9. *Let $M \in \mathbb{N}^*$. Then $BMO_{BZ2,M}(\Gamma) \subset BMO_{BZ1,M}(\Gamma)$. More precisely, for all $f \in BMO_{BZ2,M}(\Gamma)$,*

$$\|f\|_{BMO_{BZ1,M}} \lesssim \|f\|_{BMO_{BZ2,M}}.$$

Proof: Immediate consequence of Lemma 3.8. □

We want now to prove the converse inclusion, that is $BMO_{BZ1,M}(\Gamma) \subset BMO_{BZ2,M}(\Gamma)$. We begin with the next proposition, inspired from Proposition 2.6 in [15].

Proposition 3.10. *Let $M \in \mathbb{N}^*$. There exists $C > 0$ only depending on Γ and M such that for all $f \in BMO_{BZ1,M}(\Gamma)$, for all balls $B = B(x_0, \sqrt{s})$ and all integers $(a, b_1, \dots, b_M) \in \mathbb{N} \times [\![0, 2s]\!]^M$,*

$$\|P^a(I - P^{b_1}) \dots (I - P^{b_M})f\|_{L^2(B)} \leq C a_s^{\frac{d_0+1}{2}} V(B)^{\frac{1}{2}} \|f\|_{BMO_{BZ1,M}}$$

where $a_s = \max\left\{1, \frac{a}{s}\right\}$.

Remark 3.11. *We can replace $a_s^{d_0+1}$ by $a_s^{d_0+\epsilon}$ with $\epsilon > 0$ in the conclusion of the Proposition 3.10 (in this case, C depends on ϵ).*

Proof: (Proposition 3.10)

(1) Let us prove the proposition when $s \leq \min_{i \in [\![1,M]\!]} b_i$. The case where $a = 0$ is a consequence of the definition of $BMO_{BZ1,M}$ and will therefore be skipped. Let $(B_i)_{i \in I_j}$ be the covering of $C_j(B)$ provided by Corollary A.2. Then,

$$\begin{aligned}
&\|P^a(I - P^{b_1}) \dots (I - P^{b_M})f\|_{L^2(B)} \\
&\quad\lesssim \|(I - P^{b_1}) \dots (I - P^{b_M})f\|_{L^2(C_1(B))} + \sum_{j \geq 2} \exp\left(-c\frac{4^j b}{a}\right) \|(I - P^{b_1}) \dots (I - P^{b_M})f\|_{L^2(C_j(B))} \\
&\quad\leq \|(I - P^{b_1}) \dots (I - P^{b_M})f\|_{L^2(4B)} + \sum_{j \geq 2} \exp\left(-c\frac{4^j b}{a}\right) \|(I - P^{b_1}) \dots (I - P^{b_M})f\|_{L^2(2^{j+1}B)} \\
&\quad\lesssim V(4B)^{\frac{1}{2}} \|f\|_{BMO_{BZ1,M}} + \sum_{j \geq 2} \sum_{i \in I_j} \exp\left(-c\frac{4^j b}{a}\right) \|(I - P^{b_1}) \dots (I - P^{b_M})f\|_{L^2(B_i)} \\
&\quad\lesssim V(B)^{\frac{1}{2}} \|f\|_{BMO_{BZ1,M}} \left[1 + \sum_{j \geq 2} 2^{jd_0+1} \exp\left(-c\frac{4^j b}{a}\right)\right] \\
&\quad\lesssim V(B)^{\frac{1}{2}} \|f\|_{BMO_{BZ1,M}} a_s^{\frac{d_0+1}{2}}
\end{aligned} \tag{37}$$

where we use the Davies-Gaffney estimates for the first line and the doubling property for the last but one line.

(2) General case. For each $b_i < s$, write

$$(I - P^{b_i}) = (I - P^{2s}) - P^{b_i}(I - P^{2s-b_i}).$$

Hence, $P^a(I - P^{b_1}) \dots (I - P^{b_M})$ can be written as a sum of terms

$$P^{\tilde{a}}(I - P^{\tilde{b}_1}) \dots (I - P^{\tilde{b}_M})$$

where $\tilde{b}_i \in [\![s, 2s]\!]$ and $\tilde{a} \in [\![a, a + Ms]\!]$. The general case can be then deduced from the previous case.

□

Proposition 3.12. *Let $M \in \mathbb{N}^*$. There exists $C > 0$ such that for all balls B of radius $\sqrt{s}$, all integers $b \in [\![0, 2s]\!]$ and all $f \in BMO_{BZ1,M}$, one has*

$$\|(I - (I + b\Delta)^{-1})^M f\|_{L^2(B)} \leq C V(B)^{\frac{1}{2}} \|f\|_{BMO_{BZ1,M}}.$$

Proof: Let $\varphi \in L^2(\Gamma)$ supported in B. Recall that Lemma 3.2 states that φ, $(I-P^{k_1})\dots(I-P^{k_M})\varphi$ and $(I-(I+b\Delta)^{-1})^M\varphi$ are in $\mathcal{M}_0^{M,\epsilon}$ for all $\epsilon>0$. Moreover, for all $b\in\mathbb{N}$, one has

$$\begin{aligned}(I+b\Delta)^{-1}\varphi &= (1+b)^{-1}\left(I-\frac{b}{1+b}P\right)^{-1}\varphi \\ &= \sum_{k=1}^{+\infty}\left(\frac{1}{1+b}\right)\left(\frac{b}{1+b}\right)^k P^k\varphi\end{aligned}$$

where the convergence holds in $L^2(\Gamma)$. Consequently,

$$(I-(I+b\Delta)^{-1})\varphi = \sum_{k=1}^{+\infty}\left(\frac{1}{1+b}\right)\left(\frac{b}{1+b}\right)^k (I-P^k)\varphi$$

and thus,

$$(I-(I+b\Delta)^{-1})^M\varphi = \left(\frac{1}{1+b}\right)^M\sum_{k=1}^{+\infty}\left(\frac{b}{1+b}\right)^k\sum_{k_1+\dots+k_M=k}(I-P^{k_1})\dots(I-P^{k_M})\varphi$$

where the convergence still holds in $L^2(\Gamma)$.

In order to prove that the convergence holds in $\mathcal{M}_0^{M,\epsilon}$ for all $\epsilon>0$, it suffices to show that

$$S := \left(\frac{1}{1+b}\right)^M\sum_{k=1}^{+\infty}\left(\frac{b}{1+b}\right)^k\sum_{k_1+\dots+k_M=k}\|(I-P^{k_1})\dots(I-P^{k_M})\varphi\|_{\mathcal{M}_0^{M,\epsilon}} < +\infty.$$

Indeed, according to (35), one has

$$\begin{aligned}S &\lesssim \left(\frac{1}{1+b}\right)^M\sum_{k=1}^{+\infty}\left(\frac{b}{1+b}\right)^k\sum_{k_1+\dots+k_M=k}k^{\frac{\epsilon}{2}+\frac{d}{4}+1}\|\varphi\|_{L^2(B)} \\ &\lesssim \sum_{k=1}^{+\infty}\frac{(k+1)^{M+\frac{\epsilon}{2}+\frac{d}{4}}}{(1+b)^M}\left(\frac{b}{1+b}\right)^k\|\varphi\|_{L^2(B)} \\ &\lesssim b^{\frac{\epsilon}{2}+\frac{d}{4}+2}\sum_{k=1}^{+\infty}\frac{1}{(1+k)^2}\|\varphi\|_{L^2(B)} \\ &< +\infty\end{aligned}$$

where the third line comes from Lemma B.1.

For $f\in\mathcal{E}_M$, there exists $\epsilon>0$ such that $(\mathcal{M}_0^{M,\epsilon})^*$. Moreover, Lemma 3.2 states that $(I-(I+s\Delta)^{-1})^M f$ and $(I-P^{k_1})\dots(I-P^{k_M})f$ (for all $(k_1,\dots,k_M)\in\mathbb{N}^M$) are in $L^2_{loc}(\Gamma)$. As a consequence,

$$\begin{aligned}\|(I-(I+b\Delta)^{-1})^M f\|_{L^2(B)} &= \sup_{\substack{\|\varphi\|_2=1\\ \operatorname{Supp}\varphi\subset B}}|\langle f,(I-(I+b\Delta)^{-1})^M\varphi\rangle| \\ &\leq \left(\frac{1}{1+b}\right)^M\sum_{k=1}^{+\infty}\left(\frac{b}{1+b}\right)^k\sum_{k_1+\dots+k_M=k}\sup_{\substack{\|\varphi\|_2=1\\ \operatorname{Supp}\varphi\subset B}}|\langle f,(I-P^{k_1})\dots(I-P^{k_M})\varphi\rangle| \\ &= \left(\frac{1}{1+b}\right)^M\sum_{k=1}^{+\infty}\left(\frac{b}{1+b}\right)^k\sum_{k_1+\dots+k_M=k}\|(I-P^{k_1})\dots(I-P^{k_M})f\|_{L^2(B)}\end{aligned}$$

where the pairing is between $\mathcal{M}_0^{M,\epsilon}$ and its dual. Therefore

$$\begin{aligned}\|(I-(I+b\Delta)^{-1})^M f\|_{L^2(B)} &\lesssim \left(\frac{1}{1+b}\right)^M\sum_{k=1}^{+\infty}\left(\frac{b}{1+b}\right)^k\sum_{k_1+\dots+k_M=k}\|(I-P^{k_1})\dots(I-P^{k_M})f\|_{L^2(B)} \\ &\leq \left(\frac{1}{1+b}\right)^M\sum_{k=1}^{b}\left(\frac{b}{1+b}\right)^k\sum_{k_1+\dots+k_M=k}\|(I-P^{k_1})\dots(I-P^{k_M})f\|_{L^2(B)} \\ &\quad + \left(\frac{1}{1+b}\right)^M\sum_{k=b+1}^{\infty}\left(\frac{b}{1+b}\right)^k\sum_{k_1+\dots+k_M=k}\|(I-P^{k_1})\dots(I-P^{k_M})f\|_{L^2(B)} \\ &:= I_1+I_2.\end{aligned}$$

We estimate the first term with Proposition 3.10 and Lemma B.1:

$$\begin{aligned}
I_1 &\lesssim \sum_{k=1}^{b} \frac{(1+k)^{M-1}}{(1+b)^M}\left(\frac{b}{1+b}\right)^k \|f\|_{BMO_{BZ1,M}} V(B)^{\frac12} \\
&\lesssim (1+b)^{-1}\sum_{k=0}^{b-1} \|f\|_{BMO_{BZ1,M}} V(B)^{\frac12} \\
&\lesssim \|f\|_{BMO_{BZ1,M}} V(B)^{\frac12}.
\end{aligned}$$

We turn now to the estimate of the second term. One has, using Proposition 3.10 and Lemma B.1 again,

$$\begin{aligned}
I_2 &\lesssim \left(\frac{1}{1+b}\right)^M \sum_{k=b+1}^{\infty}\left(\frac{b}{1+b}\right)^k \sum_{k_1+\dots+k_M=k} \|(I-P^{k_1})\dots(I-P^{k_M})f\|_{L^2(\sqrt{\frac{k}{b}}B)} \\
&\lesssim \sum_{k=b+1}^{\infty} \frac{1}{1+k}\left(\frac{1+k}{1+b}\right)^M \left(\frac{b}{1+b}\right)^k \|f\|_{BMO_{BZ1,M}} V\left(\sqrt{\frac{k}{b}}B\right)^{\frac12} \\
&\lesssim \sum_{k=b+1}^{\infty} \frac{1}{1+k}\left(\frac{1+k}{1+b}\right)^{M+\frac{d_0}{2}+1} \left(\frac{b}{1+b}\right)^k \|f\|_{BMO_{BZ1,M}} V(B)^{\frac12} \\
&\lesssim \sum_{k=b+1}^{\infty} \frac{1}{1+k}\left(\frac{1+k}{1+b}\right)^{-1} \|f\|_{BMO_{BZ1,M}} V(B)^{\frac12} \\
&\lesssim \|f\|_{BMO_{BZ1,M}} V(B)^{\frac12},
\end{aligned}$$

where we used Proposition 1.5 for the third line. □

Corollary 3.13. *Let $M \in \mathbb{N}$. Then $BMO_{BZ1,M}(\Gamma) \subset BMO_{BZ2,M}(\Gamma)$. More precisely, for all $f \in BMO_{BZ1,M}(\Gamma)$,*

$$\|f\|_{BMO_{BZ2,M}} \lesssim \|f\|_{BMO_{BZ1,M}}.$$

Proof: Immediate consequence of Proposition 3.12. □

3.3 Duals of Hardy spaces

Proposition 3.14. *Let $\kappa \in \{1,2\}$ and $M \in \mathbb{N}^*$.*

Let ℓ be a bounded linear functional on $H^1_{BZ\kappa,M,\infty}(\Gamma)$. Then ℓ actually belongs to $BMO_{BZ\kappa,M}(\Gamma) \cap \mathcal{F}_M$ and for all $g \in \mathbb{H}^1_{BZ\kappa,M,\infty}(\Gamma)$, there holds

$$\ell(g) = \langle \ell, g\rangle \tag{38}$$

where the pairing is between $\mathcal{M}_0^{M,\epsilon}(\Gamma)$ and its dual. Moreover,

$$\|\ell\|_{BMO_{BZ\kappa,M}} \lesssim \|\ell\|_{(H^1_{BZ\kappa,M,\infty})^*}$$

Proof: Let $\kappa \in \{1,2\}$ and $M \in \mathbb{N}$.

Let ℓ in $\left[H^1_{BZ\kappa,M,\infty}(\Gamma)\right]^*$. According to Lemma 3.1, $\ell \in \bigcap_{\epsilon>0}\left[\mathcal{M}_0^{M,\epsilon}\right]^* = \mathcal{F}_M$. The following two claims

(i) $\mathbb{H}^1_{BZ\kappa,M,\infty}(\Gamma) \subset \mathcal{M}_0^{M,\epsilon}$,

(ii) $\mathbb{H}^1_{BZ\kappa,M,\infty}(\Gamma)$ is dense in $H^1_{BZ\kappa,M,\infty}(\Gamma)$,

are respectively a consequence of Lemma 3.7 and of Lemma 3.5. They imply that (38) makes sense and uniquely describes ℓ.

It remains to check the last claim, that is

$$\|\ell\|_{BMO_A} \lesssim \|\ell\|_{(H^1_{BZ\kappa,M,\infty})^*}$$

Fix $s \in \mathbb{N}^*$, a M-tuple $(s_1, \dots, s_M) \in [\![s, 2s]\!]^M$, and a ball B of radius $\sqrt{s}$. We wrote A_s for $(I-P^{s_1})\dots(I-P^{s_M})$ if $\kappa = 1$ and for $(I-(I+s\Delta)^{-1})^M$ if $\kappa = 2$.

Let $\varphi \in L^2(B)$ with norm 1. Then

$$\frac{1}{V(B)^{\frac{1}{2}}} A_s \varphi$$

is a (BZ_κ, M)-atom. Thus,

$$\left\| \frac{1}{V(B)^{\frac{1}{2}}} A_s \varphi \right\|_{H^1_{BZ\kappa,M,\infty}} \leq 1,$$

i.e.,

$$\begin{aligned} \frac{1}{V(B)^{\frac{1}{2}}} |\langle A_s \ell, \varphi\rangle| &= \frac{1}{V(B)^{\frac{1}{2}}} |\langle \ell, A_s \varphi\rangle| \\ &\lesssim \|\ell\|_{(H^1_{BZ\kappa,M,\infty})^*}. \end{aligned}$$

Lemma 3.2 provides that $A_s \ell \in L^2_{loc}(\Gamma)$. Taking the supremum over all φ supported in B, we obtain

$$\left(\frac{1}{V(B)} \sum_{x \in B} |A_s \ell(x)|^2 m(x) \right)^{\frac{1}{2}} \lesssim \|\ell\|_{(H^1_{BZ\kappa,M,\infty})^*}.$$

Finally, taking the supremum over all $s \in \mathbb{N}^*$, all M-tuples $(s_1, \dots, s_M) \in [\![s, 2s]\!]^M$ and all balls B of radius $\sqrt{s}$ leads us to the result. □

Proposition 3.15. *Let $\kappa \in \{1,2\}$ and $M \in \mathbb{N}^*$.*

Let $\epsilon > 0$ and $f \in BMO_{BZ\kappa,M}(\Gamma) \cap \mathcal{F}_M$. The linear functional given by

$$\ell(g) := \langle f, g\rangle$$

initially defined on $\mathbb{H}^1_{BZ\kappa,M,2\epsilon}(\Gamma)$, and where the pairing is between $\mathcal{M}_0^{M,\epsilon}$ and its dual, has a unique bounded extension to $H^1_{BZ\kappa,M,2\epsilon}(\Gamma)$ with

$$\|\ell\|_{(H^1_{BZ\kappa,M,2\epsilon})^*} \lesssim \|f\|_{BMO_{BZ\kappa,M}(\Gamma)}.$$

Proof: Let $\kappa \in \{1,2\}$ and $M \in \mathbb{N}^*$. In the proof, A_s will denote $(I-P^{s_1})\dots(I-P^{s_M})$ (for some $(s_1, \dots, s_M) \in [\![s, 2s]\!]^M$) or $(I-(I+s\Delta)^{-1})^M$, depending whether κ is equal to 1 or 2.

Let us prove that for every $(BZ_\kappa, M, 2\epsilon)$-molecule a, one has

$$|\langle f, a\rangle| \lesssim \|f\|_{BMO_{BZ\kappa,M}}. \tag{39}$$

Since $f \in \mathcal{F}_M$, then $f \in \left(\mathcal{M}_0^{M,\epsilon}\right)^*$. In particular, Lemma 3.2 provides that $A_s f \in L^2_{loc}(\Gamma)$. Thus, if $a = A_s b$ is a $(BZ_\kappa, M, 2\epsilon)$-molecule associated with a ball B of radius $\sqrt{s}$, we may write

$$\begin{aligned} |\langle f, a\rangle| &= \left| \sum_{x \in \Gamma} A_s f(x) b(x) m(x) \right| \\ &\leq \sum_{j \geq 1} \|A_s f\|_{L^2(C_j(B))} \|b\|_{L^2(C_j(B))} \\ &\leq \sum_{j \geq 1} 2^{-2j\epsilon} V(2^j B)^{-\frac{1}{2}} \|A_s f\|_{L^2(2^{j+1}B)} \\ &\lesssim \sum_{j \geq 1} 2^{-2j\epsilon} V(2^j B)^{-\frac{1}{2}} V(2^{j+1}B)^{\frac{1}{2}} \|f\|_{BMO_{BZ\kappa,M}} \\ &\lesssim \|f\|_{BMO_{BZ\kappa,M}}, \end{aligned}$$

where we used for the last but one line Proposition 3.10 (if $\kappa = 1$) or Proposition 3.8 and Corollary 3.9 (if $\kappa = 2$).

Our next step is to show that for every $g \in \mathbb{H}^1_{BZ\kappa,M,2\epsilon}$, we have

$$|\langle f, g\rangle| \lesssim \|g\|_{H^1_{BZ\kappa,M,2\epsilon}} \|f\|_{BMO_{BZ\kappa,M}}.$$

Indeed, let $N \in \mathbb{N}$, $(\lambda_i)_i \in [\![0, N]\!] \in \mathbb{R}^N$ and $(a_i = A_{s_i} b_i)_{i \in [\![0,N]\!]}$ a sequence of $(BZ_\kappa, M, 2\epsilon)$-molecules that satisfies $g = \sum \lambda_i a_i$ and $\sum |\lambda_i| \lesssim 2\|g\|_{H^1_{BZ\kappa,M,2\epsilon}}$, then

$$\begin{aligned}
|\ell(g)| &\leq \sum_{i=0}^{N} |\lambda_i| \, |\ell(a_i)| \\
&\lesssim \|f\|_{BMO_{BZ\kappa,M}} \sum_{i=0}^{N} |\lambda_i| \\
&\lesssim \|f\|_{BMO_{BZ\kappa,M}} \|g\|_{H^1_{BZ\kappa,M,2\epsilon}}.
\end{aligned}$$

Since $\mathbb{H}^1_{BZ\kappa,M,2\epsilon}$ is dense in $H^1_{BZ\kappa,M,2\epsilon}$, ℓ has an unique bounded extension that satisfies

$$\|\ell\|_{(H^1_{BZ\kappa,M,2\epsilon})^*} \lesssim \|f\|_{BMO_{BZ\kappa,M}}.$$

$\square$

Proposition 3.16. *Let $\kappa \in \{1, 2\}$ and $M \in \mathbb{N}^*$.*
Let $f \in BMO_{BZ\kappa,M}(\Gamma)$ and let $\epsilon > 0$ such that $f \in (\mathcal{M}_0^{M,\epsilon}(\Gamma))^*$. The linear functional given by

$$\ell(g) := \langle f, g \rangle$$

initially defined on $\mathbb{H}^1_{BZ\kappa,M,\infty}(\Gamma)$ which is a dense subset of $\mathcal{M}_0^{M,\epsilon}$, and where the pairing is that between $\mathcal{M}_0^{M,\epsilon}$ and its dual, has a unique extension to $H^1_{BZ\kappa,M,\infty}(\Gamma)$ with

$$\|\ell\|_{(H^1_{BZ\kappa,M,\infty})^*} \lesssim \|f\|_{BMO_{BZ\kappa,M}}.$$

Proof: Same proof than Proposition 3.15 with obvious modifications. The only difference is: in Proposition 3.15, $\epsilon > 0$ is given by the Hardy space $H^1_{BZ\kappa,M,2\epsilon}$ and in Proposition 3.16, $\epsilon > 0$ is given by the functional $f \in \mathcal{E}_M$. $\square$

We turn now to the proof of Theorem 1.35.

Proof: Let $\kappa \in \{1, 2\}$ and $M \in \mathbb{N}^*$.

Proposition 3.14 and Corollary 3.16 provide the continuous embeddings

$$(H^1_{BZ\kappa,M,\infty})^* \hookrightarrow BMO_{BZ\kappa,M} \cap \mathcal{F}_M \hookrightarrow BMO_{BZ\kappa,M} \hookrightarrow (H^1_{BZ\kappa,M,\infty})^*.$$

As a consequence, $BMO_{BZ\kappa,M}$ is the dual space of $H^1_{BZ\kappa,M,\infty}$ and is actually included in $\mathcal{F}_M$.

Besides, Propositions 3.14 and 3.16 yield, for any $\epsilon > 0$

$$(H^1_{BZ\kappa,M,\infty})^* \hookrightarrow BMO_{BZ\kappa,M} \cap \mathcal{F}_M \hookrightarrow (H^1_{BZ\kappa,M,\epsilon})^*.$$

Since the inclusion $(H^1_{BZ\kappa,M,\epsilon})^* \hookrightarrow (H^1_{BZ\kappa,M,\infty})^*$ is obvious, we obtain that $BMO_{BZ\kappa,M} \cap \mathcal{F}_M = BMO_{BZ\kappa,M}$ is also the dual space of $H^1_{BZ\kappa,M,\epsilon}$.

The last claim of the Theorem, that is for a fixed $M \in \mathbb{N}^*$, the spaces $H^1_{BZ\kappa,M,\epsilon}(\Gamma)$ for $\kappa \in \{1, 2\}$ and $\epsilon \in (0, +\infty]$ are all equivalent, is only a consequence of the proposition 3.17 below. Indeed, for $m \in \mathbb{N}^*$ and $\kappa \in \{1, 2\}$, the inclusion $H^1_{BZ\kappa,M,\epsilon} \subset H^1_{BZ\kappa,M,\eta}$ when $0 < \eta < \epsilon \leq +\infty$ is obvious and then Proposition 3.17 yields the equality between the spaces $H^1_{BZ\kappa,M,\epsilon}$ for $\epsilon \in (0, +\infty]$, together with the equivalence of norms. It remains to check that, for example, $H^1_{BZ1,M,\infty} \subset H^1_{BZ2,M,1}$. For this, notice first that similarly to (36), for a (BZ_1, M)-atom a associated with $s \in \mathbb{N}^*$, $(s_1, \dots, s_m) \in [\![s, 2s]\!]^M$, a ball B of radius $\sqrt{s}$ and a function $b \in L^2(B)$, one has

$$\begin{aligned}
a &= (I - P^{s_1}) \dots (I - P^{s_M}) b \\
&= (I - (I + s\Delta)^{-1})^M \prod_{i=1}^{M} \left[\frac{s_i}{s} Q_{s_i} + (I - P^{s_i}) \right] b.
\end{aligned}$$

We have to check that $\prod_{i=1}^{M} \left[\frac{s_i}{s} Q_{s_i} + (I - P^{s_i}) \right] b$ satisfies, up to a multiplicative constant, the estimates given by (ii) of the definition of a $(BZ_2, M, 1)$-molecule. This calculus, which is a straightforward consequence of the Gaffney estimates provided by Proposition 2.6, is left to the reader. $\square$

Proposition 3.17. *If $(E, \|.\|_E)$ and $(F, \|.\|_F)$ are two Banach spaces with the same dual $(G, \|.\|_G)$ and moreover if we have the continuous inclusion $E \subset F$, then $E = F$ with equivalent norms.*

Proof: Let T be the linear operator defined by

$$T : e \in E \mapsto e \in F.$$

T is bounded and its adjoint T^* is

$$T^* : g \in G \mapsto g \in G,$$

that is the identity on G. Theorem 4.15 in [23] implies that $E = F$, and then, by the open mapping theorem, we deduce that the norm of E is dominated by the norm of F. □

4 Inclusions between Hardy spaces

4.1 $H^1_{BZ1,M,\epsilon} \cap L^2 \subset E^1_{quad,\beta}$: the case of functions

Proposition 4.1. *Let $\epsilon > 0$, $M \in (\frac{d_0}{4}, +\infty) \cap \mathbb{N}$ and $\beta > 0$. Then $H^1_{BZ1,M,\epsilon}(\Gamma) \cap L^2(\Gamma) \subset E^1_{quad,\beta}(\Gamma)$ and*

$$\|f\|_{H^1_{quad,\beta}} \lesssim \|f\|_{H^1_{BZ1,M,\epsilon}}$$

Proof: Let $f \in H^1_{BZ1,M,\epsilon} \cap L^2(\Gamma)$. Then there exist $(\lambda_i)_{i\in\mathbb{N}} \in \ell^1$ and $(a_i)_{i\in\mathbb{N}}$ a sequence of $(BZ1, M, \epsilon)$-molecules such that $f = \sum \lambda_i a_i$ and

$$\sum_{i\in\mathbb{N}} |\lambda_i| \simeq \|f\|_{H^1_{BZ1,M,\epsilon}}.$$

First, since $\|P^k\|_{1\to 1} \leq 1$ for all $k \in \mathbb{N}$, the operators Δ^β and then $\Delta^\beta P^{l-1}$ are L^1-bounded for $\beta > 0$ (see [11]). Consequently,

$$\Delta^\beta P^{l-1} \sum_{i\in\mathbb{N}} \lambda_i a_i = \sum_{i\in\mathbb{N}} \lambda_i \Delta^\beta P^{l-1} a_i.$$

Since the space Γ is discrete, the L^1-convergence implies the pointwise convergence, that is, for all $x \in \Gamma$,

$$\begin{aligned} \left| \Delta^\beta P^{l-1} \sum_{i\in\mathbb{N}} \lambda_i a_i(x) \right| &= \left| \sum_{i\in\mathbb{N}} \lambda_i \Delta^\beta P^{l-1} a_i(x) \right| \\ &\leq \sum_{i\in\mathbb{N}} |\lambda_i| \left| \Delta^\beta P^{l-1} a_i(x) \right|. \end{aligned}$$

From here, the estimate

$$\|L_\beta f\|_{L^1} = \left\| L_\beta \sum_{i\in\mathbb{N}} \lambda_i a_i \right\|_{L^1} \lesssim \sum_{i\in\mathbb{N}} |\lambda_i| \|L_\beta a_i\|_{L^1}$$

is just a consequence of the generalized Minkowski inequality.

It remains to prove that there exists a constant C such that for all $(BZ1, M, \epsilon)$-molecules a, one has

$$\|L_\beta a\|_{L^1} \leq C. \tag{40}$$

Let $s \in \mathbb{N}^*$, $(s_1, \dots, s_M) \in [\![s, 2s]\!]^M$ and a ball B associated with the molecule a. By Hölder inequality and the doubling property, we may write

$$\|L_\beta a\|_{L^1} \lesssim \sum_{j=1}^{\infty} V(2^j B)^{\frac{1}{2}} \|L_\beta a\|_{L^2(C_j(B))}. \tag{41}$$

We will estimate now each term $\|L_\beta a\|_{L^2(C_j(B))}$.

The result is then a consequence of Lemma 2.14 which can be reformulated as follows

$$\|L_\beta (I - P^{s_1}) \dots (I - P^{s_M})[f \mathbb{1}_F]\|_{L^2(E)} \leq C_M \left(1 + \frac{d(E, F)^2}{s}\right)^{-M} \|f\|_{L^2(F)}. \tag{42}$$

Notice that

$$d(C_k(B), C_j(B)) \simeq \begin{cases} 0 & \text{if } |j-k| \leq 1 \\ 2^j\sqrt{s} & \text{if } k \leq j-2 \\ 2^k\sqrt{s} & \text{if } k \geq j+2 \end{cases}.$$

Thus,

$$\begin{aligned}
\|L_\beta a\|_{L^2(C_j(B))} &\leq \sum_{k\geq 1} \|L_\beta (I-P^{s_1})\dots(I-P^{s_M})[b\mathbb{1}_{C_k(B)}]\|_{L^2(C_j(B))} \\
&\lesssim \sum_{k\leq j-2} 4^{-jM}\|b\|_{L^2(C_k(B))} + \sum_{k=j-1}^{j+1} \|b\|_{L^2(C_k(B))} + \sum_{k\geq j+2} 4^{-kM}\|b\|_{L^2(C_k(B))} \\
&\lesssim \sum_{k\leq j-2} 4^{-jM}2^{-\epsilon k}V(2^kB)^{-\frac{1}{2}} + 2^{-\epsilon j}V(2^jB)^{-\frac{1}{2}} + \sum_{k\geq j+2} 4^{-kM}2^{-\epsilon j}V(2^kB)^{-\frac{1}{2}} \\
&\lesssim 2^{-\bar{\epsilon} j}V(2^jB)^{-\frac{1}{2}}
\end{aligned}$$

where $\bar{\epsilon} = \min\{\epsilon, 2M - \frac{d_0}{2}\}$.
As a consequence, one has

$$\begin{aligned}
\|L_\beta a\|_{L^1} &\lesssim \sum_{j\geq 1} 2^{-\bar{\epsilon} j}\left(\frac{V(2^jB)}{V(2^jB)}\right)^{\frac{1}{2}} \\
&< +\infty.
\end{aligned}$$

□

Proposition 4.2. *Let* (Γ,μ) *satisfying* (UE), $M \in \mathbb{N}^*$, $\epsilon > 0$ *and* $\beta > 0$. *Then* $H^1_{BZ1,M}(\Gamma) \cap L^2(\Gamma) \subset E^1_{quad,\beta}(\Gamma)$ *and*

$$\|f\|_{H^1_{quad,\beta}} \lesssim \|f\|_{H^1_{BZ1,M,\epsilon}}$$

Proof: As in the proof of Proposition 4.1, it remains to check that for all $(BZ1, M, \epsilon)$-molecules $a = (I-P^{s_1})\dots(I-P^{s_M})b$ associated with $s \in \mathbb{N}$, $(s_1,\dots,s_M)$ and $B = B(x_B, r_B)$, one has

$$\sum_{j=1}^{\infty} V(2^jB)^{\frac{1}{2}}\|L_\beta a\|_{L^2(C_j(B))} \lesssim 1.$$

The case $j = 1$ follows from the L^2-boundedness of L_β and of $(I-P^s)^M$, thus

$$\|L_\beta a\|_{L^2(C_j(B))} \lesssim \|a\|_{L^2} \lesssim \frac{1}{V(B)^{\frac{1}{2}}}.$$

For the case $j \geq 2$, we introduce $\tilde{C}_j(B)$ defined by

$$\tilde{C}_j(B) = \bigcup_{1\leq k\leq j-2} C_k(B).$$

Check that $\tilde{C}_j(B)$, $C_j(B)$, and x_B satisfy (28), since $d(\tilde{C}_j(B), C_j(B)) \gtrsim 2^j r_B$. Thus, Lemma 2.18 yields

$$\begin{aligned}
\|L_\beta a\|_{L^2(C_j(B))} &\leq \|L_\beta(I-P^s)^M[b\mathbb{1}_{\tilde{C}_j(B)}]\|_{L^2(C_j(B))} + \|L_\beta(I-P^s)^M[b\mathbb{1}_{\Gamma\backslash\tilde{C}_j(B)}]\|_{L^2(C_j(B))} \\
&\lesssim \frac{4^{-jM}}{V(x_B, 2^jr_B)^{\frac{1}{2}}}\|b\|_{L^1(\tilde{C}_j(B))} + \|b\|_{L^2(\Gamma\backslash\tilde{C}_j(B))} \\
&\lesssim \frac{2^{-j\bar{\epsilon}}}{V(2^jB)^{\frac{1}{2}}}
\end{aligned}$$

where $\bar{\epsilon} = \min\{2M, \epsilon\}$. Summing in $j \geq 1$ ends the proof. □

4.2 $H^1_{BZ2,M+\frac{1}{2},\epsilon} \cap H^2 \subset E^1_{quad,\beta}$: the case of 1-forms

Proposition 4.3. *Let $\epsilon > 0$, $M \in (\frac{d_0}{4} - \frac{1}{2}, +\infty) \cap \mathbb{N}$. Then $H^1_{BZ2,M+\frac{1}{2}}(T_\Gamma) \cap H^2(T_\Gamma) \subset E^1_{quad,\frac{1}{2}}(T_\Gamma)$ and*

$$\|f\|_{H^1_{quad,\frac{1}{2}}} \lesssim \|f\|_{H^1_{BZ2,M+\frac{1}{2},\epsilon}}.$$

Proof: Let $F \in H^1_{BZ2,M+\frac{1}{2},\epsilon}(T_\Gamma) \cap H^2(T_\Gamma)$. Then there exist $(\lambda_i)_{i\in\mathbb{N}} \in \ell^1$ and $(a_i)_{i\in\mathbb{N}}$ a sequence of $(BZ2, M + \frac{1}{2}, \epsilon)$-molecules such that $F = \sum \lambda_i a_i$ and

$$\sum_{i\in\mathbb{N}} |\lambda_i| \simeq \|f\|_{H^1_{BZ1,M,\epsilon}}.$$

First, by L^1-boundedness of the operators P and d^* (see Proposition 1.32) and by the Minkowski inequality, one has

$$\begin{aligned}
\|L_{\frac{1}{2}}\Delta^{-\frac{1}{2}} d^* F\|_{L^1} &= \sum_{x\in\Gamma} m(x) \left(\sum_{l\geq 1} \sum_{y\in B(x,\sqrt{l})} m(y) |P^{l-1} d^* F(y)|^2 \right)^{\frac{1}{2}} \\
&= \sum_{x\in\Gamma} m(x) \left(\sum_{l\geq 1} \sum_{y\in B(x,\sqrt{l})} m(y) |P^{l-1} d^* \sum_{i\in\mathbb{N}} \lambda_i a_i(y)|^2 \right)^{\frac{1}{2}} \\
&\lesssim \sum_{i\in\mathbb{N}} |\lambda_i| \sum_{x\in\Gamma} m(x) \left(\sum_{l\geq 1} \sum_{y\in B(x,\sqrt{l})} m(y) |P^{l-1} d^* a_i(y)|^2 \right)^{\frac{1}{2}}.
\end{aligned}$$

It remains to prove that there exists a constant C such that for all $(BZ2, M + \frac{1}{2}, \epsilon)$-molecules a, one has

$$\sum_{x\in\Gamma} m(x) \left(\sum_{l\geq 1} \sum_{y\in B(x,\sqrt{l})} m(y) |P^{l-1} d^* a_i(y)|^2 \right)^{\frac{1}{2}} \lesssim 1. \tag{43}$$

Let $a = d\Delta^{-\frac{1}{2}}(I - (I + s\Delta)^{-1})^{M+\frac{1}{2}} b$ be a $(BZ_2, M + \frac{1}{2}, \epsilon)$-molecule associated with $s \in \mathbb{N}^*$ and the ball B. Since $d^* d\Delta^{-\frac{1}{2}} = \Delta^{\frac{1}{2}}$, (43) becomes

$$\|L_{\frac{1}{2}}(I - (I + s\Delta)^{-1})^{M+\frac{1}{2}} b\| \lesssim 1.$$

We end the proof as we did for Proposition 4.1, using Lemma 2.16 instead of Lemma 2.14. □

Proposition 4.4. *Let (Γ, μ) satisfying* (UE). *Let $\epsilon > 0$, $M \in \mathbb{N}$. Then $H^1_{BZ2,M+\frac{1}{2}}(T_\Gamma) \cap H^2(T_\Gamma) \subset E^1_{quad,\frac{1}{2}}(T_\Gamma)$ and*

$$\|f\|_{H^1_{quad,\frac{1}{2}}} \lesssim \|f\|_{H^1_{BZ2,M+\frac{1}{2},\epsilon}}.$$

Proof: We begin the proof as the one of Proposition 4.3. We end the proof as Proposition 4.2 instead of Proposition 4.1, using Lemma 2.19 instead of Lemma 2.16. □

4.3 $E^1_{quad,\beta} \subset H^1_{BZ2,M,\epsilon} \cap L^2$: the case of functions

In this paragraph, we will need a few results on tents spaces (see [7], [25], [19]). However, we need in our proofs some "discrete" tent spaces, defined below:

Definition 4.5. *For $x \in \Gamma$, we recall*

$$\gamma(x) = \{(y, k) \in \Gamma \times \mathbb{N},\, d(x, y)^2 \leq k\}$$

and for a set $O \subset \Gamma$, we define

$$\hat{O} = \{(y, k) \in \Gamma \times \mathbb{N},\, d(y, O^c)^2 > k\}.$$

For a function F defined on $\Gamma \times \mathbb{N}$, consider for all $x \in \Gamma$

$$\mathcal{A}F(x) = \left(\sum_{(y,k)\in\gamma(x)} \frac{1}{k+1} \frac{m(y)}{V(x,\sqrt{k+1})} |F(y,k)|^2 \right)^{\frac{1}{2}}.$$

For $p \in [1,+\infty)$, the tent space $T_2^p(\Gamma)$ is defined as the space of functions F on $\Gamma \times \mathbb{N}$ for which $\mathcal{A}F \in L^p(\Gamma)$, and is outfitted with the norm $\|F\|_{T_2^p} = \|\mathcal{A}F\|_{L^p}$ (the space T_2^p is then complete).

Definition 4.6. *A function A on $\Gamma \times \mathbb{N}$ is said to be a T_2^1-atom if there exists a ball $B \subset \Gamma$ such that A is supported in $\hat{B}$ and*

$$\|A\|^2_{T_2^2} := \sum_{(x,k)\in\hat{B}} \frac{m(x)}{k+1} |A(x,k)|^2 \leq \frac{1}{V(B)}.$$

Proposition 4.7. *For every element $F \in T_2^1(\Gamma)$, there exist a scalar sequence $(\lambda_i)_{i\in\mathbb{N}} \subset \ell^1$ and a sequence of T_2^1-atoms $(A_i)_{i\in\mathbb{N}}$ such that*

$$F = \sum_{i=0}^{+\infty} \lambda_i A_i \qquad \text{in } T_2^1(\Gamma). \tag{44}$$

Moreover,

$$\sum_{i\geq 0} |\lambda_i| \simeq \|F\|_{T_2^1}$$

where the implicit constants only depend on the constant in (DV). *Finally, if $F \in T_2^1(\Gamma) \cap T_2^2(\Gamma)$, then the decomposition* (44) *also converges in $T_2^2(\Gamma)$.*

Proof: This proof is analogous to the one of Theorem 1.1 in [25] and of Theorem 4.10 in [19] with obvious modifications. □

We introduce the functional $\pi_{\eta,\beta} :\ T_2^2(\Gamma) \to L^2(\Gamma)$ defined for any real $\beta > 0$ and any integer $\eta \geq \beta$ by

$$\pi_{\eta,\beta} F(x) = \sum_{l\geq 1} \frac{c_l^\eta}{l^\beta} \left[\Delta^{\eta-\beta} (I+P)^\eta P^{l-1} F(.,l-1) \right](x)$$

where $\sum_{l\geq 1} c_l^\eta z^{l-1}$ is the Taylor series of the function $(1-z)^{-\eta}$.

Lemma 4.8. *The operator $\pi_{\eta,\beta}$ is bounded from $T_2^2(\Gamma)$ to $L^2(\Gamma)$.*

Proof: Let $g \in L^2(\Gamma)$. Then, for all $F \in T_2^2(\Gamma)$,

$$\begin{aligned}
\langle \pi_{\eta,\beta} F, g \rangle &= \sum_{l\geq 1} \frac{c_l^\eta}{l^\beta} \left\langle \Delta^{\eta-\beta}(I+P)^\eta P^{l-1} F(.,l), g \right\rangle \\
&= \sum_{l\geq 1} \frac{c_l^\eta}{l^\beta} \left\langle F(.,l-1), \Delta^{\eta-\beta}(I+P)^\eta P^{l-1} g \right\rangle \\
&\leq \sum_{l\geq 1} \frac{c_l^\eta}{l^\beta} \|F(.,l-1)\|_{L^2} \|\Delta^{\eta-\beta}(I+P)^\eta P^{l-1} g\|_{L^2} \\
&\leq \left(\sum_{l\geq 1} \frac{1}{l} \|F(.,l-1)\|^2_{L^2} \right)^{\frac{1}{2}} \left(\sum_{l\geq 1} l^{1-2\beta} (c_l^\eta)^2 \|\Delta^{\eta-\beta}(I+P)^\eta P^{l-1} g\|^2_{L^2} \right)^{\frac{1}{2}} \\
&\lesssim \|F\|_{T_2^2} \|(I+P)^\eta g\|_{L^2} \\
&\lesssim \|F\|_{T_2^2} \|g\|_{L^2}
\end{aligned}$$

where the last but one line comes from the L^2-boundedness of Littlewood-Paley functionals (since $l^{1-2\beta}(c_l^\eta)^2 \simeq l^{2(\eta-\beta)-1}$, see [17], Lemma B.1). □

Lemma 4.9. *Suppose that A is a $T_2^1(\Gamma)$-atom associated with a ball $B \subset \Gamma$. Then for every $M \in \mathbb{N}^*$, $\beta > 0$ and $\epsilon \in (0, +\infty)$, there exist an integer $\eta = \eta_{M,\beta,\epsilon}$ and a uniform constant $C_{M,\beta,\epsilon} > 0$ such that $C_{M,\beta,\epsilon}^{-1} \pi_{\eta,\beta}(A)$ is a (BZ_2, M, ϵ)-molecule associated with the ball B.*

Proof: Let $\eta = \lceil \frac{d_0}{4} + \frac{\epsilon}{2} + \beta \rceil + M + 1$, that is the only integer such that

$$\eta \geq \frac{d_0}{4} + \frac{\epsilon}{2} + \beta + M + 1 > \eta - 1.$$

Let A be a T_2^1-atom associated with a ball B of radius r. We write

$$a := \pi_{\eta,\beta}(A) = (I - (I + r^2\Delta)^{-1})^M b$$

where

$$b := \sum_{l \geq 1} \frac{c_l^\eta}{l^\beta} \left(\frac{I + r^2 \Delta}{r^2} \right)^M \Delta^{\eta - \beta - M} (I + P)^\eta P^{l-1} A(., l-1)$$

Let us check that a is a (BZ_2, M, ϵ)-molecule associated with B, up to multiplication by some harmless constant $C_{M,\epsilon}$. First, one has, for all $g \in L^2(4\eta B)$,

$$\begin{aligned}
|\langle b, g \rangle| &\leq \sum_{m=0}^{M} \frac{c_m}{r^{2(M-m)}} \sum_{l \geq 1} \frac{c_l^\eta}{l^\beta} \left| \left\langle \Delta^{\eta-\beta-M+m} (I+P)^\eta P^{l-1} A(., l-1), g \right\rangle \right| \\
&= \sum_{m=0}^{M} \frac{c_m}{r^{2(M-m)}} \sum_{l \geq 1} \frac{c_l^\eta}{l^\beta} \left| \left\langle A(., l-1), \Delta^{\eta-\beta-M+m} (I+P)^\eta P^{l-1} g \right\rangle \right| \\
&\lesssim \sum_{m=0}^{M} \frac{1}{r^{2(M-m)}} \sum_{l \geq 1} l^{\eta-\beta-1} \|A(., l-1)\|_{L^2(B)} \|\Delta^{\eta-\beta-M+m} (I+P)^\eta P^{l-1} g\|_{L^2(B)} \\
&\lesssim \sum_{m=0}^{M} \frac{1}{r^{2(M-m)}} \|A\|_{T_2^2} \left(\sum_{l=1}^{r^2} l^{2(\eta-\beta)-1} \|\Delta^{\eta-\beta-M+m} (I+P)^\eta P^{l-1} g\|_{L^2(B)}^2 \right)^{\frac{1}{2}} \\
&\lesssim \sum_{m=0}^{M} \|A\|_{T_2^2} \left(\sum_{l=1}^{r^2} l^{2(\eta-\beta-M+m)-1} \|\Delta^{\eta-\beta-M+m} (I+P)^\eta P^{l-1} g\|_{L^2(B)}^2 \right)^{\frac{1}{2}} \\
&\lesssim \|A\|_{T_2^2} \sum_{m=0}^{M} \|G_{\eta-\beta-M+m} (I+P)^\eta g\|_{L^2} \\
&\lesssim \|A\|_{T_2^2} \|(I+P)^\eta g\|_{L^2} \\
&\lesssim \frac{1}{V(B)^{\frac{1}{2}}} \|g\|_{L^2}
\end{aligned}$$

where we used the L^2-boundedness of the quadratic Littlewood-Paley functional for the last but one line (see [17], [1]).

Let $j > \log_2(\eta) + 1$ and $g \in L^2(C_j(B))$. Since $\mathrm{Supp}(I + P)^\eta g \in C_{j,\eta}(B) = \{x \in \Gamma, d(x, C_j(B)) \leq \eta\}$ and

$d(C_{j,\eta}(B),B) \gtrsim 2^j r,$

$$\begin{aligned}
|\langle b,g\rangle| &\lesssim \sum_{m=0}^{M} \frac{1}{r^{2(M-m)}}\|A\|_{T_2^2}\left(\sum_{l=1}^{r^2} l^{2(\eta-\beta)-1}\|\Delta^{\eta-\beta-M+m}(I+P)^\eta P^{l-1}g\|^2_{L^2(B)}\right)^{\frac{1}{2}}\\
&\lesssim \sum_{m=0}^{M} r^{2(\eta-\beta-M-1)}\|A\|_{T_2^2}\left(\sum_{l=1}^{r^2} l^{2(1+m)-1}\|\Delta^{\eta-\beta-M+m}(I+P)^\eta P^{l-1}g\|^2_{L^2(B)}\right)^{\frac{1}{2}}\\
&\lesssim r^{2(\eta-\beta-M-1)}\|A\|_{T_2^2}\sum_{m=0}^{M}\left\|G_{1+m}\Delta^{\eta-\beta-M-1}(I+P)^\eta g\right\|_{L^2(B)}\\
&\lesssim \frac{r^{2(\eta-\beta-M-1)}}{(4^j r^2)^{\eta-\beta-M-1}}\|A\|_{T_2^2}\,\|(I+P)^\eta g\|_{L^2}\\
&\lesssim 2^{-j(\frac{d_0}{2}+\epsilon)}\|A\|_{T_2^2}\,\|g\|_{L^2}\\
&\lesssim \frac{2^{-j\epsilon}}{V(2^jB)^{\frac{1}{2}}}\,\|g\|_{L^2}
\end{aligned}$$

where we used Lemma 2.15 for the last but two line and Proposition 1.5 for the last one. We conclude that, up to multiplication by some harmless constant, b is a (BZ_2,M,ϵ)-molecule. □

Proposition 4.10. *Let $M\in\mathbb{N}^*$, $\epsilon>0$ and $\beta>0$. Then $E^1_{quad,\beta}(\Gamma)\subset H^1_{BZ2,M,\epsilon}(\Gamma)\cap L^2(\Gamma)$ and*

$$\|f\|_{H^1_{BZ2,M,\epsilon}} \lesssim \|f\|_{H^1_{quad,\beta}}.$$

Proof: Let $f\in E^1_{quad,\beta}(\Gamma)$. We set

$$F(.,l)=[(l+1)\Delta]^\beta P^l f.$$

By definition of $H^1_{quad,\beta}(\Gamma)$, one has that $F\in T^1_2(\Gamma)$. Moreover, since $f\in L^2(\Gamma)$, L^2-boundedness of Littlewood-Paley functionals (see [4], [17]) yields that $F\in T^2_2(\Gamma)$. Thus, according to Lemma 4.7, there exist a numerical sequence $(\lambda_i)_{i\in\mathbb{N}}$ and a sequence of T^1_2-atoms $(A_i)_{i\in\mathbb{N}}$ such that

$$F=\sum_{i=0}^{\infty}\lambda_i A_i \qquad \text{in } T^1_2(\Gamma) \text{ and } T^2_2(\Gamma)$$

and

$$\sum_{i\in\mathbb{N}}|\lambda_i| \lesssim \|F\|_{T^1_2} = \|f\|_{H^1_{quad,\beta}}.$$

Choose η as in Lemma 4.9. Using Corollary 2.3, since $f\in L^2(\Gamma)$,

$$\begin{aligned}
f &= \pi_{\eta,\beta}F(.,l)\\
&= \sum_{i=0}^{+\infty}\lambda_i\pi_{\eta,\beta}(A_i)
\end{aligned} \tag{45}$$

where the sum converges in $L^2(\Gamma)$. According to Lemma 4.9, $\pi_{\eta,\beta}(A_i)$ are molecules and then (45) would provide a (M,ϵ)-representation of f if the convergence held in $L^1(\Gamma)$. By uniqueness of the limit, it remains to prove that $\sum\lambda_i\pi_{\eta,\beta}(A_i)$ converges in L^1. Indeed,

$$\begin{aligned}
\sum_{i\in\mathbb{N}}|\lambda_i|\,\|\pi_{\eta,\beta}(A_i)\|_{L^1} &\lesssim \sum_{i\in\mathbb{N}}|\lambda_i|\\
&< +\infty
\end{aligned}$$

where the first line comes from Proposition 2.7 and the second one from the fact that $(\lambda_i)_{i\in\mathbb{N}}\in\ell^1(\mathbb{N})$. □

4.4 $E^1_{quad,\beta} \subset H^1_{BZ2,M+\frac{1}{2},\epsilon} \cap H^2$: the case of 1-forms

Lemma 4.11. *Suppose that A is a $T^1_2(\Gamma)$-atom associated with a ball $B \subset \Gamma$. Let $M \in \mathbb{N}$ and $\epsilon > 0$, there exist an integer $\eta = \eta_{M,\epsilon}$ and a uniform constant $C_{M,\epsilon} > 0$ such that $C^{-1}_{M,\epsilon} d\Delta^{-\frac{1}{2}}\pi_{\eta,\frac{1}{2}}(A)$ is a $(BZ_2, M+\frac{1}{2}, \epsilon)$-molecule associated with the ball B.*

Proof: Let $\eta = \lceil \frac{d_0}{4} + \frac{\epsilon}{2} \rceil + M + 2$. We will also write t for $\lceil \frac{d_0}{4} + \frac{\epsilon}{2} \rceil \in \mathbb{N}^*$.

Let A be a T^1_2-atom associated with a ball B of radius r. We write

$$a := d\Delta^{-\frac{1}{2}}\pi_{\eta,\frac{1}{2}}(A) = r^{2M+1} d\Delta^M (I + r^2\Delta)^{-M-\frac{1}{2}} b$$

where

$$\begin{aligned} b &:= \sum_{l\geq 1} \frac{c^\eta_l}{\sqrt{l}} \left(\frac{I + r^2\Delta}{r^2}\right)^{M+\frac{1}{2}} \Delta^{\eta-1-M}(I+P)^\eta P^{l-1} A(.,l-1) \\ &= \sqrt{\frac{r^2}{1+r^2}} \sum_{k=0}^\infty a_k \left(\frac{r^2}{1+r^2}\right)^k \sum_{l\geq 1} \frac{c^\eta_l}{\sqrt{l}} \left(\frac{I+r^2\Delta}{r^2}\right)^{M+1} \Delta^{1+t}(I+P)^\eta P^{l+k-1} A(.,l-1) \end{aligned} \tag{46}$$

where $\sum a_k z^k$ is the Taylor serie of the function $(1-z)^{-\frac{1}{2}}$ (cf (25)).

Let us check that a is a $(BZ_2, M+\frac{1}{2}, \epsilon)$-molecule associated with B, up to multiplication by some harmless constant $C_{M,\epsilon}$.

Let $g \in L^2(4\eta B)$. One has with the first equality in (46),

$$\begin{aligned} |\langle b, g\rangle| &\leq r^{-2M-1} \sum_{l\geq 1} \frac{c^\eta_l}{\sqrt{l}} \left| \left\langle A(.,l-1), \left(I + r^2\Delta\right)^{M+\frac{1}{2}} \Delta^{1+t}(I+P)^\eta P^{l-1} g \right\rangle \right| \\ &\lesssim r^{-2M-1} \sum_{l\geq 1} \frac{c^\eta_l}{\sqrt{l}} \|A(.,l-1)\|_{L^2(B)} \| \left(I + r^2\Delta\right)^{M+\frac{1}{2}} \Delta^{1+t}(I+P)^\eta P^{l-1} g\|_{L^2} \\ &\lesssim \|A\|_{T^2_2} r^{-2M-1} \left(\sum_{l=1}^{r^2} l^{2(\eta-1)} \| \left(I + r^2\Delta\right)^{M+\frac{1}{2}} \Delta^{1+t}(I+P)^\eta P^{l-1} g\|^2_{L^2} \right)^{\frac{1}{2}} \\ &\lesssim \|A\|_{T^2_2} r^{-2M-1} \left(\sum_{l=1}^{r^2} l^{2(1+t+M)} \| \left(I + r^2\Delta\right)^{M+\frac{1}{2}} \Delta^{1+t}(I+P)^\eta P^{l-1} g\|^2_{L^2} \right)^{\frac{1}{2}} \\ &\lesssim \|A\|_{T^2_2} \|(I+P)^\eta g\|_{L^2} \\ &\lesssim \|A\|_{T^2_2} \|g\|_{L^2} \end{aligned}$$

where we use that the functionals $g \mapsto r^{-2M-1} \left(\sum_{l=1}^{r^2} l^{2(1+t+M)} | \left(I + r^2\Delta\right)^{M+\frac{1}{2}} \Delta^{1+t} P^{l-1} g|^2 \right)^{1/2}$ are L^2-bounded uniformly in r. Indeed, since $(-1) \notin Sp(P)$, functional calculus provides, for some $a > -1$,

$$\begin{aligned} \| \left(I + r^2\Delta\right)^{M+\frac{1}{2}} \Delta^{1+t} P^{l-1} g\|^2_{L^2} &= \int_a^1 (1 + r^2(1-\lambda))^{2M+1} (1-\lambda)^{2(1+t)} \lambda^{2(l-1)} dE_{gg}(\lambda) \\ &\lesssim \int_a^1 \left[1 + r^{2(2M+1)}(1-\lambda)^{2M+1}\right] (1-\lambda)^{2(1+t)} \lambda^{2(l-1)} dE_{gg}(\lambda). \end{aligned}$$

Thus,

$$
\begin{aligned}
& r^{-2(2M+1)}\sum_{l=1}^{r^2} l^{2(1+t+M)}\| \left(I+r^2\Delta\right)^{M+\frac12}\Delta^{1+t}P^{l-1}g\|_{L^2}^2 \\
&\qquad \lesssim \int_a^1 (1-\lambda)^{2(1+t)}\sum_{l=1}^{r^2} l^{2(1+t)-1}\lambda^{2(l-1)}dE_{gg}(\lambda)+\int_a^1(1-\lambda)^{2(1+t+M)+1}\sum_{l=1}^{r^2} l^{2(1+t+M)}\lambda^{2(l-1)}dE_{gg}(\lambda) \\
&\qquad \lesssim \int_a^1 (1-\lambda)^{2(1+t)}\sum_{l=1}^{\infty} l^{2(1+t)-1}\lambda^{2(l-1)}dE_{gg}(\lambda)+\int_a^1(1-\lambda)^{2(1+t+M)+1}\sum_{l=1}^{\infty} l^{2(1+t+M)}\lambda^{2(l-1)}dE_{gg}(\lambda) \\
&\qquad \lesssim \int_a^1 \frac{(1-\lambda)^{2(1+t)}}{(1-\lambda^2)^{2(1+t)}}dE_{gg}(\lambda)+\int_a^1\frac{(1-\lambda)^{2(1+t+M)+1}}{(1-\lambda^2)^{2(1+t+M)+1}}dE_{gg}(\lambda) \\
&\qquad = \int_a^1 \left[(1+\lambda)^{-2(1+t)}+(1+\lambda)^{-2(1+t+M)-1}\right]dE_{gg}(\lambda) \\
&\qquad \lesssim \int_a^1 dE_{gg}(\lambda)=\|g\|_{L^2}^2
\end{aligned}
$$

where the third inequality comes from the fact that $l^{\xi-1}\sim c_l^{\xi}$ (see Lemma B.1 in [17]).

Let $j>\log_2(\eta)+1$ and $g\in L^2(C_j(B))$. One has $d(C_{j,\eta}(B),B)\gtrsim 2^j r$ (cf Lemma 4.9). The second identity in (46) provides

$$
\begin{aligned}
|\langle b,g\rangle| &\le \sum_{m=0}^{M+1}\frac{c_m}{r^{2(M+1-m)}}\sqrt{\frac{r^2}{1+r^2}}\sum_{k=0}^{\infty}a_k\left(\frac{r^2}{1+r^2}\right)^k\sum_{l\ge1}\frac{c_l^\eta}{\sqrt{l}}\left|\left\langle A(.,l-1),\Delta^{1+t+m}(I+P)^\eta P^{l+k-1}g\right\rangle\right| \\
&\lesssim \sum_{m=0}^{M+1}\frac{c_m}{r^{2(M+1-m)}}\sum_{k=0}^{\infty}a_k\left(\frac{r^2}{1+r^2}\right)^k\sum_{l\ge1}\frac{c_l^\eta}{\sqrt{l}}\|A(.,l-1)\|_{L^2(B)}\|\Delta^{1+t+m}(I+P)^\eta P^{l+k-1}g\|_{L^2(B)} \\
&\lesssim \|A\|_{T_2^2}\sum_{m=0}^{M+1}\frac{1}{r^{2(M+1-m)}}\sum_{k=0}^{\infty}a_k\left(\frac{r^2}{1+r^2}\right)^k\left(\sum_{l=1}^{r^2}l^{2(\eta-1)}\|\Delta^{1+t+m}(I+P)^\eta P^{l+k-1}g\|_{L^2(B)}^2\right)^{\frac12} \\
&\lesssim \|A\|_{T_2^2}\|(I+P)^\eta g\|_{L^2}\sum_{m=0}^{M+1}r^{2(\eta-M-2+m)}\sum_{k=0}^{\infty}a_k\left(\frac{r^2}{1+r^2}\right)^k\left(\sum_{l=1}^{r^2}\frac{e^{-c\frac{4^jr^2}{l+k}}}{(l+k)^{2(1+t+m)}}\right)^{\frac12} \\
&\lesssim \|A\|_{T_2^2}\|g\|_{L^2}\sum_{m=0}^{M+1}r^{2(t+m)}\sum_{k=0}^{\infty}a_k\left(\frac{r^2}{1+r^2}\right)^k\left(\sum_{l=1}^{\infty}\frac{e^{-c\frac{4^jr^2}{l+k}}}{(l+k)^{2(1+t+m)}}\right)^{\frac12} \\
&\lesssim \|A\|_{T_2^2}\|g\|_{L^2}\sum_{m=0}^{M+1}r^{2(t+m)+1}\frac{1}{\sqrt{1+r^2}}\sum_{k=0}^{\infty}a_k\left(\frac{r^2}{1+r^2}\right)^k\frac{1}{(4^jr^2)^{t+m+\frac12}} \\
&\lesssim \|A\|_{T_2^2}\|g\|_{L^2}\sum_{m=0}^{M+1}\frac{r^{2(t+m)+1}}{(4^jr^2)^{t+m+\frac12}}(1+r^2(1-1))^{-\frac12} \\
&\lesssim 2^{-j(2t+1)}\|A\|_{T_2^2}\|g\|_{L^2} \\
&\lesssim \frac{2^{-j\epsilon}}{V(2^jB)}\|g\|_{L^2}
\end{aligned}
$$

where we used the estimate (GUE) for the forth line. □

Proposition 4.12. *Let $M\in\mathbb{N}$ and $\epsilon>0$. Then $E^1_{quad,\frac12}(T_\Gamma)\subset H^1_{BZ2,M+\frac12}(T_\Gamma)\cap H^2(T_\Gamma)$ and*

$$\|G\|_{H^1_{BZ2,M+\frac12,\epsilon}}\lesssim\|G\|_{H^1_{quad,\frac12}}\qquad\forall G\in E^1_{quad,\frac12}(T_\Gamma)$$

Proof: Let $G \in E^1_{quad,\frac{1}{2}}(T_\Gamma)$. We set

$$F(.,l) = \sqrt{l+1}P^l d^*G.$$

By definition of $H^1_{quad,\frac{1}{2}}(T_\Gamma)$, one has that $F \in T^1_2(\Gamma)$. Moreover, Proposition 1.32 yields that $\Delta^{-\frac{1}{2}}d^*G \in L^2(G)$ and therefore $F \in T^2_2(\Gamma)$ with the L^2-boundedness of Littlewood-Paley functionals.

Thus, according to Lemma 4.7, there exist a scalar sequence $(\lambda_i)_{i\in\mathbb{N}} \in \ell^1(\mathbb{N})$ and a sequence of T^1_2-atoms $(A_i)_{i\in\mathbb{N}}$ such that

$$F = \sum_{i=0}^{\infty} \lambda_i A_i \qquad \text{in } T^1_2(\Gamma) \text{ and in } T^2_2(\Gamma)$$

and

$$\sum_{i\in\mathbb{N}} |\lambda_i| \lesssim \|F\|_{T^1_2} = \|G\|_{H^1_{quad,\frac{1}{2}}}.$$

Choose η as in Lemma 4.9. Using Lemma 2.3, since $\Delta^{-\frac{1}{2}}d^*G \in L^2(\Gamma)$,

$$\begin{aligned}\Delta^{-\frac{1}{2}}d^*G &= \pi_{\eta,\frac{1}{2}}F(.,l) \\ &= \sum_{i=0}^{+\infty} \lambda_i \pi_{\eta,\frac{1}{2}}(A_i)\end{aligned}$$

where the sum converges in $L^2(\Gamma)$. Recall that $d\Delta^{-1}d^* = Id_{H^2(T_\Gamma)}$. Moreover, $d\Delta^{-\frac{1}{2}}$ is bounded from $L^2(\Gamma)$ to $L^2(T_\Gamma)$ (see Proposition 1.32). Then

$$G = \sum_{i=0}^{+\infty} \lambda_i d\Delta^{-\frac{1}{2}}\pi_{\eta,\frac{1}{2}}(A_i) \tag{47}$$

where the sum converges in $L^2(T_\Gamma)$. According to Lemma 4.11, $d\Delta^{-\frac{1}{2}}\pi_{M,\frac{1}{2}}(A_i)$ are $(BZ2, M+\frac{1}{2},\epsilon)$-molecules and then (47) would provide a $(BZ2, M+\frac{1}{2},\epsilon)$-representation of f if the convergence held in $L^1(\Gamma)$. By uniqueness of the limit, it remains to prove that $\sum \lambda_i d\Delta^{-\frac{1}{2}}\pi_{\eta,\frac{1}{2}}(A_i)$ converges in L^1. Indeed,

$$\begin{aligned}\sum_{i\in\mathbb{N}} |\lambda_i| \left\| d\Delta^{-\frac{1}{2}}\pi_{\eta,\frac{1}{2}}(A_i)\right\|_{L^1(T_\Gamma)} &\lesssim \sum_{i\in\mathbb{N}} |\lambda_i| \\ &< +\infty\end{aligned}$$

where the first line comes from Corollary 2.12 and the second one because $(\lambda_i)_{i\in\mathbb{N}} \in \ell^1(\mathbb{N})$. □

4.5 Proof of Theorems 1.36, 1.38 and 1.39

Proof: (Theorem 1.36)

Let $\beta > 0$, $M \in \mathbb{N}^* \cap (\frac{d_0}{4}, +\infty)$ and $\epsilon > 0$. Propositions 4.1 and 4.10 yield the continuous embeddings

$$H^1_{BZ1,M,\epsilon}(\Gamma) \cap L^2(\Gamma) \subset E^1_{quad,\beta}(\Gamma) \subset H^1_{BZ2,M,\epsilon}(\Gamma) \cap L^2(\Gamma).$$

However, Theorem 1.34 states that $H^1_{BZ1,M,\epsilon}(\Gamma) = H^1_{BZ2,M,\epsilon(\Gamma)}$. Thus, we deduce

$$H^1_{BZ1,M,\epsilon}(\Gamma) \cap L^2(\Gamma) = E^1_{quad,\beta}(\Gamma) = H^1_{BZ2,M,\epsilon}(\Gamma) \cap L^2(\Gamma) \tag{48}$$

with equivalent norms. In particular, $E^1_{quad,\beta}(\Gamma) \subset L^1(\Gamma)$.

Let us now prove that the completion of $E^1_{quad,\beta}(\Gamma)$ in L^1 exists. To that purpose, it is enough (see Proposition 2.2 in [2]) to check that, for all Cauchy sequences $(f_n)_n$ in $E^1_{quad,\beta}(\Gamma)$ that converges to 0 in $L^1(\Gamma)$, $f_n \to 0$ for the $\|.\|_{H^1_{quad,\beta}}$ norm. Equivalent norms in (48) implies that $(f_n)_n$ is a Cauchy sequence in $H^1_{BZ\kappa,M,\epsilon}(\Gamma)$ that converges to 0 in $L^1(\Gamma)$. Since $H^1_{BZ\kappa,M,\epsilon}(\Gamma)$ is complete, it follows that $f_n \to g$ for some $g \in H^1_{BZ\kappa,M,\epsilon}(\Gamma)$, but then also for the L^1-norm, which entails that $g = 0$. Thus, $f_n \to 0$ for the norm $H^1_{BZ\kappa,M,\epsilon}(\Gamma)$ and so for the norm $\|.\|_{H^1_{quad,\beta}}$ (the norms being equivalent on $E^1_{quad,\beta}(\Gamma)$).

Therefore, the completion $H^1_{quad,\beta}(\Gamma)$ of $E^1_{quad,\beta}(\Gamma)$ exists and is defined by

$$H^1_{quad,\beta}(\Gamma) = \{f \in F, \text{ there exists } (f_n)_n \text{ Cauchy sequence in } E^1_{quad,\beta}(\Gamma) \text{ such that } f_n \to f \text{ in } L^1(\Gamma)\}.$$

The fact that $H^1_{quad,\beta}(\Gamma) = H^1_{BZ\kappa,M\epsilon}(\Gamma)$ is then a straightforward consequence of (48) and the fact that the space $H^1_{BZ\kappa,M\epsilon}(\Gamma) \cap L^2(\Gamma)$ is dense in $H^1_{BZ\kappa,M\epsilon}(\Gamma)$. □

Proof: (Theorem 1.38)

Let $M \in \mathbb{N} \cap (\frac{d_0}{4} - \frac{1}{2}, +\infty)$ and $\epsilon > 0$. Propositions 4.3 and 4.12 yield the continuous embeddings

$$H^1_{BZ2,M+\frac{1}{2},\epsilon}(T_\Gamma) \cap H^2(T_\Gamma) \subset E^1_{quad,\frac{1}{2}}(T_\Gamma) \subset H^1_{BZ2,M+\frac{1}{2},\epsilon}(T_\Gamma) \cap H^2(T_\Gamma),$$

from which we deduce the equality of the two spaces, with equivalent norms.

Since $\mathbb{H}^1_{BZ2,M+\frac{1}{2},\epsilon}(T_\Gamma)$ is dense in $H^1_{BZ2,M+\frac{1}{2},\epsilon}(T_\Gamma) \subset L^1(T_\Gamma)$ and is included in $H^1_{BZ2,M+\frac{1}{2},\epsilon}(T_\Gamma) \cap H^2(\Gamma)$, it follows that $H^1_{BZ2,M+\frac{1}{2},\epsilon}(T_\Gamma)$ is the completion in $L^1(T_\Gamma)$ of $H^1_{BZ2,M+\frac{1}{2},\epsilon}(T_\Gamma) \cap H^2(\Gamma)$ and thus also of $E^1_{quad,\frac{1}{2}}(T_\Gamma)$ with the same arguments than those used in the proof of Theorem 1.36.

Moreover, notice that if $F \in H^2(T_\Gamma)$,

$$F \in E^1_{quad,\beta}(T_\Gamma) \Longleftrightarrow \Delta^{-\frac{1}{2}} d^* F \in E^1_{quad,\beta}(\Gamma).$$

Indeed, the implication $\Delta^{-\frac{1}{2}} d^* F \in E^1_{quad,\beta}(\Gamma) \Rightarrow F \in E^1_{quad,\beta}(T_\Gamma)$ is obvious, and the converse is due to Proposition 1.32. As said in Theorem 1.36, the spaces $E^1_{quad,\beta}(\Gamma)$ are all equivalent once $\beta > 0$; and so are the spaces $E^1_{quad,\beta}(T_\Gamma)$. Consequently, for all $\beta > 0$, the completion of $E^1_{quad,\beta}(T_\Gamma)$ in $L^1(T_\Gamma)$ exists and is the same as the one of $E^1_{quad,\frac{1}{2}}(T_\Gamma)$. $\square$

Proof: (Theorem 1.39)

Just use Proposition 4.2 instead of Proposition 4.1 (in the proof of Theorem 1.36), and Proposition 4.4 instead of Proposition 4.3 (in the proof of Theorem 1.38). $\square$

Let us state and prove now item *b*) of Remark 1.41. We first introduce $E^1_{BZ\kappa,M,\epsilon}(\Gamma)$ defined by

$$E^1_{BZ\kappa,M,\epsilon}(\Gamma) := \left\{ f \in L^2(\Gamma),\ \sum_{j=0}^{\infty} \lambda_j a_j \text{ is a molecular } (BZ_\kappa, M, \epsilon)\text{-representation of } f \text{ and the sum converges in } L^2(\Gamma) \right\}$$

and outfitted with the norm

$$\|f\|_{E^1_{BZ\kappa,M,\epsilon}} = \inf \left\{ \sum_{i\in\mathbb{N}} |\lambda_i|,\ \sum_{j=0}^{\infty} \lambda_j a_j \text{ is a molecular } (BZ_\kappa, M, \epsilon)\text{-representation of } f \text{ and the sum converges in } L^2(\Gamma) \right\}.$$

In the same way, we define $E^1_{BZ2,M+\frac{1}{2},\epsilon}(T_\Gamma)$ by

$$E^1_{BZ2,M+\frac{1}{2},\epsilon}(T_\Gamma) := \left\{ f \in H^2(T_\Gamma),\ \sum_{j=0}^{\infty} \lambda_j a_j \text{ is a mol. } (BZ_2, M + \frac{1}{2}, \epsilon)\text{-representation of } f \text{ and the sum converges in } L^2(T_\Gamma) \right\}$$

and we equipped it with the norm

$$\|f\|_{E^1_{BZ\kappa,M+\frac{1}{2},\epsilon}} = \inf \left\{ \sum_{i\in\mathbb{N}} |\lambda_i|,\ \sum_{j=0}^{\infty} \lambda_j a_j \text{ is a mol. } (BZ_2, M + \frac{1}{2}, \epsilon)\text{-representation of } f \text{ and the sum converges in } L^2(T_\Gamma) \right\}.$$

Corollary 4.13. *Let Γ be a weighted graph satisfying* (DV) *and* (LB).

(i) If $\kappa \in \{1,2\}$, $\epsilon \in (0,+\infty)$ and $M \in \mathbb{N}^* \cap (\frac{d_0}{4}, +\infty)$, then

$$E^1_{BZ\kappa,M,\epsilon}(\Gamma) = H^1_{BZ\kappa,M,\epsilon}(\Gamma) \cap L^2(\Gamma) = E^1_{quad,1}(\Gamma)$$

with equivalent norms. As a consequence, the completion of $E^1_{BZ\kappa,M,\epsilon}(\Gamma)$ in $L^1(\Gamma)$ exists and is equal to $H^1(\Gamma) = H^1_{BZ\kappa,M,\epsilon}(\Gamma)$.

(ii) If $\epsilon \in (0,+\infty)$ and $M \in \mathbb{N} \cap (\frac{d_0}{4} - \frac{1}{2}, +\infty)$, then

$$E^1_{BZ2,M+\frac{1}{2},\epsilon}(T_\Gamma) = H^1_{BZ2,M+\frac{1}{2},\epsilon}(T_\Gamma) \cap L^2(\Gamma) = E^1_{quad,\frac{1}{2}}(T_\Gamma)$$

with equivalent norms. As a consequence, the completion of $E^1_{BZ2,M,\epsilon}(T\Gamma)$ in $L^1(T_\Gamma)$ exists and is equal to $H^1(T_\Gamma) = H^1_{BZ2,M+\frac{1}{2},\epsilon}(T_\Gamma)$.

(iii) If the Markov kernel $p(x,y)$ satisfies the pointwise gaussian bound (UE)*, then M can be choosen arbitrarily in $\mathbb{N}^*$ in (i) and in $\mathbb{N}$ in (ii).*

Proof: The proof consists in noticing, as the proofs show, that the (BZ_κ, M, ϵ) (resp. $(BZ_2, M+\frac{1}{2}, \epsilon)$) representation of $f \in E^1_{quad,1}(\Gamma)$ (resp. $F \in E^1_{quad,\frac{1}{2}}(T_\Gamma)$) constructed in Proposition 4.10 (resp. 4.12) also converges in $L^2(\Gamma)$ (resp. $L^2(T_\Gamma)$).

Therefore, we proved in Propositions 4.10 and 4.12 that

$$E^1_{quad,1}(\Gamma) \subset E^1_{BZ\kappa,M,\epsilon}(\Gamma) \subset H^1_{BZ\kappa,M,\epsilon}(\Gamma) \cap L^2(\Gamma)$$

and

$$E^1_{quad,\frac{1}{2}}(T_\Gamma) \subset E^1_{BZ2,M+\frac{1}{2},\epsilon}(T_\Gamma) \subset H^1_{BZ2,M+\frac{1}{2},\epsilon}(T_\Gamma) \cap L^2(T_\Gamma).$$

We end then the proof as in Theorems 1.36, 1.38 and 1.39. □

A A covering lemma

Lemma A.1. *Let B a ball of radius $r \in \mathbb{N}^*$ and $\alpha \geq 1$. There exists a collection of pairwise disjoint balls $(B_i)_{i\in I_\alpha}$ of radius r such that*

$$\bigcup_{i\in I_\alpha} B_i \subset \alpha B \subset \bigcup_{i\in I_\alpha} 3B_i.$$

Proof: It is a classical fact and we provide a proof for completeness. Let B be a ball of radius r and of center x_0. Let $(B_i)_{i\in I_\alpha}$ be a set of disjoint balls included in αB and of radius r. Assume that $(B_i)_{i\in I_\alpha}$ is maximal, that is, for every ball B_0 of radius r, either B_0 is not included in αB, or there exist $i \in I_\alpha$ such that $B_0 \cap B_i \neq \emptyset$. Let us prove that

$$\alpha B \subset \bigcup_{i\in I_\alpha} 3B_i. \tag{49}$$

Let $x \in \alpha B$ and let us prove that the ball $B(x,2r)$ intersects one of the B_i's. Assume the opposite. There exists a path $x_0, x_1, \dots, x_{n-1}, x$ joining x_0 to x and of length $n = d(x,x_0) < \alpha r$. Then the balls $B(x_{\max\{0,n-r\}}, r)$ is included in $B(x,2r)$ and in αB, that is the set $(B_i)_{i\in I_\alpha}$ is not maximal. By contradiction, there exists $i \in I_\alpha$ such that $B(x,2r) \cap B_i \neq \emptyset$, that implies $x \in 3B_i$. □

Corollary A.2. *There exist $M \in \mathbb{N}$ and $C > 0$ such that for all balls B of radius r and all $j \geq 1$, there exists a covering $(B_i)_{i\in I_j}$ of $C_j(B)$ such that*

(i) each ball B_i is of radius r,

(ii) the covering is included in $\tilde{C}_j := C_{j-1}(B) \cup C_j(B) \cup C_{j+1}(B)$ (with the convention $C_0(B) = \emptyset$), that is

$$\bigcup_{i\in I_j} B_i \subset \tilde{C}_j$$

(iii) each point is covered by at most M balls B_i.

(iv) the number of balls $\# I_j$ is bounded by $C2^{j(d_0+1)}$

Proof: Let B be a ball of radius r and $j \geq 1$. Notice that (iv) is a consequence of the three first points. Indeed,

$$\begin{aligned}
\# I_j &= \frac{1}{V(2^j B)} \sum_{i\in I_j} V(2^j B) \\
&\leq \frac{1}{V(2^j B)} \sum_{i\in I_j} V(2^{j+3} B_i) \\
&\lesssim \frac{2^{j(d_0+1)}}{V(2^j B)} \sum_{i\in I_j} V(B_i) \\
&\leq M 2^{j(d_0+1)} \frac{1}{V(2^j B)} V(2^{j+2} B) \\
&\lesssim 2^{j(d_0+1)}.
\end{aligned}$$

where the second line is a consequence of (i) and (ii), the third one holds thanks to Proposition 1.5, and the forth one is due to (ii) and (iii).

Let us now prove the first three conclusions of the corollary.

Assume that $r \in \{1,2\}$. Then the collection of balls $(B(x,r))_{x\in C_j(B)}$ satisfies (i), (ii) and (iii). Indeed, only (iii) for $r=2$ is not obvious, but is a consequence of the uniform local finiteness of Γ.

Assume now that $r \geq 3$. Let $s \in \left[\frac{r}{5}, \frac{r}{3}\right] \cap \mathbb{N}$. By Lemma A.1 (with $\alpha = 2^{j+1}\frac{s}{r}$), there exists a collection $(\tilde{B}_i)_{i\in I_\alpha}$ of balls of radius s such that

$$\bigcup_{i\in I_\alpha} \tilde{B}_i \subset 2^{j+1}B \subset \bigcup_{i\in I_\alpha} 3\tilde{B}_i.$$

We set

$$I_j = \{i \in I_\alpha,\ 3\tilde{B}_i \cap C_j(B) \neq \emptyset\}$$

and then $B_i = \frac{r}{s}\tilde{B}_i$. Let us check that the collection of balls $(B_i)_{i\in I_j}$ satisfies the conclusions of the corollary. (i) is a consequence of the construction. (ii) is true since

$$\bigcup_{i\in I_j} B_i \subset \{x \in \Gamma,\ d(x, C_j(B)) < 2s\}.$$

For the point (iii), define for $x \in \Gamma$,

$$I_x = \{i \in I_j, B(x,s) \cap B_i \neq \emptyset\}.$$

Since all $\tilde{B}_i$ are disjoints, one has then

$$\sum_{i\in I_x} V(\tilde{B}_i) \leq V(x, r+s) \leq V(x, 6s).$$

However, notice that $B(x,6s) \subset V(12\tilde{B}_i)$ for all $i \in I_x$. Hence, with the doubling property,

$$V(x,6s) \gtrsim \sum_{i\in I_x} V(12\tilde{B}_i) \gtrsim \sum_{i\in I_x} V(x,6s)$$

and therefore, $\# I_x \lesssim 1$. $\square$

B Exponential decay of some functions

Lemma B.1. *For all $m \in [0,+\infty)$, there exists $C_m, c > 0$ such that for all $t \geq 0$ and $k \in \mathbb{N}$, one has*

$$\left(\frac{1+k}{1+t}\right)^m \left(\frac{t}{1+t}\right)^k \leq C_m \exp\left(-c\frac{k}{1+t}\right).$$

Proof: First check that the function

$$\varphi(t) \in \mathbb{R}_+^* \mapsto \left(1 - \frac{1}{1+t}\right)^{1+t}$$

satisfies $0 < \varphi(t) < 1$ for all $t > 0$ and $\lim_{t\to\infty} \varphi(t) = e^{-1} < 1$. Then there exists $c > 0$ such that $\varphi(t) \in (0, e^{-c})$ for all $t > 0$. From here, one has

$$\begin{aligned}\left(\frac{1+k}{1+t}\right)^m \left(\frac{t}{1+t}\right)^k &\leq \left(1+\frac{k}{1+t}\right)^m \left(\frac{t}{1+t}\right)^k \\ &= \left(1+\frac{k}{1+t}\right)^m \exp\left(\frac{k}{1+t}\ln\varphi(t)\right) \\ &\leq \left(1+\frac{k}{1+t}\right)^m \exp\left(-c\frac{k}{1+t}\right) \\ &\leq C_m \exp\left(-\frac{c}{2}\frac{k}{1+t}\right).\end{aligned}$$

$\square$

References

[1] C. Arhancet and C. Le Merdy. Dilation of Ritt operators on L^p-spaces. 2011. Available as http://arxiv.org/abs/1106.1513.

[2] P. Auscher, A. McIntosh, and A. J. Morris. Calderòn reproducing formulas and applications to Hardy spaces. Available as http://arxiv.org/abs/1304.0168.

[3] P. Auscher, A. McIntosh, and E. Russ. Hardy spaces of differential forms and Riesz transforms on Riemannian manifolds. *J. Geom. Anal.*, 18(1):192–248, 2008.

[4] N. Badr and E. Russ. Interpolation of Sobolev spaces, Littlewood-Paley inequalities and Riesz transforms on graphs. *Publ. Mat.*, 53:273–328, 2009.

[5] F. Bernicot and J. Zhao. New abstract Hardy spaces. *J. Funct. Anal.*, 255:1761–1796, 2008.

[6] T. A. Bui and X. T. Duong. Hardy spaces associted to the discrete laplacians on graphs and boundedness of singular integrals. *Trans. Amer. Math. Soc.*, 255:1761–1796, 2014.

[7] R. R. Coifman, Y. Meyer, and E. M. Stein. Some new function spaces and their applications to harmonic analysis. *J. Funct. Analysis*, 62:304–335, 1985.

[8] R. R. Coifman and G. Weiss. Extensions of Hardy spaces and their use in analysis. *Bull. Amer. Math. Soc.*, 83(4):569–645, 1977.

[9] T. Coulhon and A. Grigor'yan. Random walks on graphs with regular volume growth. *Geom. Funct. Anal.*, 8(4):656–701, 1998.

[10] T. Coulhon, A. Grigor'yan, and F. Zucca. The discrete integral maximum principle and its applications. *Tohoku Math. J.*, 57:559–587, 2005.

[11] T. Coulhon and L. Saloff-Coste. Puissances d'un opérateur régularisant. *Ann. Inst. H. Poincaré Probab. Statist.*, 26(3):419–436, 1990.

[12] T. Delmotte. Parabolic Harnack inequality and estimates of Markov chains on graphs. *Revista Matemàtica Iberoamericana*, 15(1):181–232, 1999.

[13] N. Dungey. A note on time regularity for discrete time heat kernels. *Semigroups forum*, 72(3):404–410, 2006.

[14] X. T. Duong and L. X. Yan. Duality of Hardy and BMO spaces associated with operators with heat kernel bounds. *J. Amer. Math. Soc.*, 18(4):943–973, 2005.

[15] X. T. Duong and L. X. Yan. New function spaces of BMO type, the John-Niremberg inequality, interpolation, and applications. *Comm. Pure Appl. Math.*, 58(10):1375–1420, 2005.

[16] C. Fefferman and E. M. Stein. H^p spaces of several variables. *Acta Math.*, 129(3-4):137–193, 1972.

[17] J. Feneuil. Littlewood-Paley functionals on graphs. *Math. Nachr.*, to appear. Available as http://fr.arxiv.org/abs/1404.1353.

[18] Alexander Grigoryan. Analysis on graphs. *Lecture Notes, University Bielefeld*, 2009.

[19] S. Hofmann, G. Lu, D. Mitrea, M. Mitrea, and L. Yan. Hardy spaces associated to non-negative self-adjoint operators satisfying Davies-Gaffney estimates. *Mem. Amer. Math. Soc.*, 214(1007):vi+78, 2011.

[20] S. Hofmann and J. M. Martell. L^p bounds for Riesz transforms and square roots associated to second order elliptic operators. *Publ. Mat.*, 47(2):497–515, 2003.

[21] S. Hofmann and S. Mayboroda. Hardy and BMO spaces associated to divergence form elliptic operators. *Math. Ann.*, 344:37–116, 2009.

[22] Y. Meyer. *Ondelettes et opérateurs. II.* Actualités Mathématiques. [Current Mathematical Topics]. Hermann, Paris, 1990. Opérateurs de Calderón-Zygmund. [Calderón-Zygmund operators].

[23] W Rudin. *Functional Analysis.* International Series in Pure and Applied Mathematics. McGraw-Hill, Inc., New York, second edition, 1991.

[24] E. Russ. Riesz tranforms on graphs for $1 \leq p \leq 2$. *Math. Scand.*, 87(1):133–160, 2000.

[25] E. Russ. The atomic decomposition for tent spaces on spaces of homogeneous type. In *CMA/AMSI Research Symposium "Asymptotic Geometric Analysis, Harmonic Analysis, and Related Topics"*, volume 42 of *Proc. Centre Math. Appl. Austral. Nat. Univ.*, pages 125–135. Austral. Nat. Univ., Canberra, 2007.

[26] E. M. Stein and G. Weiss. On the theory of harmonic functions of several variables. I. The theory of H^p-spaces. *Acta Math.*, 103:25–62, 1960.